TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 315, Number 2, October 1989

GEOMETRIC QUANTIZATION AND THE UNIVERSAL ENVELOPING ALGEBRA OF A NILPOTENT LIE GROUP

NIELS VIGAND PEDERSEN

ABSTRACT. We study geometric quantization in connection with connected nilpotent Lie groups. First it is proved that the quantization map associated with a (real) polarized coadjoint orbit establishes an isomorphism between the space of polynomial quantizable functions and the space of polynomial quantized operators. Our methods allow noninductive proofs of certain basic facts from Kirillov theory. It is then shown how the quantization map connects with the universal enveloping algebra. This is the main result of the paper. Finally we show how one can explicitly compute global canonical coordinates on coadjoint orbits, and that this can be done simultaneously on all orbits contained in a given stratum of what we call "the fine $\mathscr{F}$-stratification of the dual of the Lie algebra". This is a generalization of a result of M. Vergne about simultaneous canonical coodinates for orbits in general position.

INTRODUCTION

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let O be the coadjoint orbit through the element $g \in \mathfrak{g}^*$. Further, let $\mathfrak{h}$ be a *real* polarization at g, and let F be the corresponding G-invariant polarization of the symplectic manifold O. In this situation the space $\mathscr{E}_F^1(O)$ of smooth F-quantizable functions on O is well defined (cf., e.g. [9]). Let H_0 be the analytic subgroup corresponding to $\mathfrak{h}$, and set $H = G_g H_0$, where G_g is the stabilizer of g in G, and suppose that there exists a unitary character $\chi : H \to \mathbf{T}$ such that $\chi(\exp X) = e^{i\langle g, X\rangle}$ for all $X \in \mathfrak{h}$. This means that the orbit O is integral. We have in [9] defined the space $\mathscr{B}^1(G, \chi)$ of quantized operators. This space can be considered as the set of all differential operators of order at most one in the homogeneous line bundle with base G/H defined by χ. The main result of [9] was that the quantization map

$$\delta_\chi : \mathscr{E}_F^1(O) \to \mathscr{B}^1(G, \chi)$$

is a Lie algebra *isomorphism.*

Suppose now that G is nilpotent and simply connected. Then the space of polynomial functions $\mathscr{P}(O)$ on O is well defined, and therefore also the space $\mathscr{P}_F^1(O)$ $(\subset \mathscr{E}_F^1(O))$ of polynomial quantizable functions. Similarly we can speak

Received by the editors April 11, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22E27.

of the space $\mathscr{A}^1(G,\chi)$ ($\subset \mathscr{B}^1(G,\chi)$) of polynomial quantized operators. In the first part of the paper we show that the image of the space of polynomial quantizable functions $\mathscr{P}_F^1(O)$ by the quantization map δ_χ is equal to the space of polynomial quantized operators of order at most one $\mathscr{A}^1(G,\chi)$ (Theorem 2.4.1), i.e., δ_χ defines a Lie algebra isomorphism

$$\delta_\chi : \mathscr{P}_F^1(O) \to \mathscr{A}^1(G,\chi).$$

Combined with Pukanszky's noninductive proof of the Kirillov character formula [10], our methods can be used to give noninductive proofs of two other basic facts from the Kirillov theory, namely that one can associate a well-defined class of irreducible representations of G with each coadjoint orbit O of G, and that the image of the universal enveloping algebra by the differential of such a representation can be identified with the algebra $\mathscr{A}(\mathbf{R}^n)$ of all polynomial differential operators on some space $\mathbf{R}^n$ (the Weyl algebra); in fact, we carry out this program in §5 (5.4.6).

Let now $\pi_\chi = \mathrm{ind}_{H\uparrow G}\chi$ be the induced representation of χ. Then π_χ is a representative of the class of irreducible representations associated with O. Denoting by $d\pi_\chi$ the differential of π_χ considered as a representation of $U(\mathfrak{g}_\mathbf{C})$, the universal enveloping algebra of the complexification $\mathfrak{g}_\mathbf{C}$ of $\mathfrak{g}$, we can for each $u \in U(\mathfrak{g}_\mathbf{C})$ consider the operator $d\pi_\chi(u)$ as an element of the space $\mathscr{B}(G,\chi)$ corresponding to *all* differential operators in the line bundle mentioned above, and, in fact, it is easily seen that $d\pi_\chi(u)$ belongs to the space $\mathscr{A}(G,\chi)$ corresponding to all *polynomial* differential operators. Now from the theorem of Kirillov alluded to above, [5] (and its extension in [2]), we have that actually $d\pi_\chi(U(\mathfrak{g}_\mathbf{C})) = \mathscr{A}(G,\chi)$. Let us now set $J_F^1(O)$ to be the set of elements in $U(\mathfrak{g}_\mathbf{C})$ which are mapped by $d\pi_\chi$ to differential operators of order at most one, i.e., $J_F^1(O) = \{u \in U(\mathfrak{g}_\mathbf{C}) \mid d\pi_\chi(u) \in \mathscr{A}^1(G,\chi)\}$ (it is easily seen that the definition of $J_F^1(O)$ only depends on F). It then follows from Kirillov's theorem that $d\pi_\chi(J_F^1(O)) = \mathscr{A}^1(G,\chi)$. If we set $I(O) = \ker(d\pi_\chi)$ we can factor $d\pi_\chi$ through $I(O)$ to get an isomorphism

$$d\tilde{\pi}_\chi : J_F^1(O)/I(O) \to \mathscr{A}^1(G,\chi).$$

Now from the result above the quantization map

$$\delta_\chi : \mathscr{P}_F^1(O) \to \mathscr{A}^1(G,\chi)$$

is also an isomorphism. This means that there is a Lie algebra isomorphism $\tilde{\Phi} : \mathscr{P}_F^1(O) \to J_F^1(O)/I(O)$ which makes the following diagram commutative:

$$\begin{tikzcd} \mathscr{P}_F^1(O) \arrow[rr, "\tilde{\Phi}"] \arrow[dr, "\delta_\chi"'] & & J_F^1(O)/I(O) \arrow[dl, "d\tilde{\pi}_\chi"] \\ & \mathscr{A}^1(G,\chi) & \end{tikzcd}$$

In §3 we consider the problem of finding an explicit description of $\tilde{\Phi}$. This problem has probably no noncomplicated solution for arbitrary polarizations. It turns out, however, that if $\mathfrak{h}$ is the Vergne polarization associated with a given Jordan-Hölder basis in $\mathfrak{g}$, then there is a simple explicit description of $\tilde{\Phi}$ in terms of the symmetrization map (Theorem 3.1.2). We use this to show how to construct a representation π of G on $L^2(\mathbf{R}^{d/2})$ associated with a given orbit O for which not only $d\pi$ is very easily computable in geometric terms, but for which also a cross-section of the map $d\pi$ from $U(\mathfrak{g}_{\mathbf{C}})$ to the algebra $\mathscr{A}(\mathbf{R}^{d/2})$ of all polynomial differential oparators on $\mathbf{R}^{d/2}$ is very explicitly computable (Theorem 3.2.1). In §6 we give an example of such computations.

In several previous papers I have made essential use of a stratification, due to Pukanszky, of the dual of a nilpotent or solvable Lie algebra into finitely many strata. This stratification depends only on the choice of a Jordan-Hölder sequence $\mathscr{F}$ in $\mathfrak{g}$. We shall call it the *coarse $\mathscr{F}$-stratification* of $\mathfrak{g}^*$. For a long time I have known that a finer stratification can be useful, e.g., when considering characters of solvable Lie groups (cf. [8]). Here this finer stratification is defined in the nilpotent case (§4). We shall call it the *fine $\mathscr{F}$-stratification* of $\mathfrak{g}^*$. Let $\mathscr{E}$ denote the index set of this finer stratification, and let Ω_ε denote the stratum corresponding to $\varepsilon \in \mathscr{E}$. We then have $\mathfrak{g}^* = \bigcup_{\varepsilon\in\mathscr{E}} \Omega_\varepsilon$. In §5 we show that there exists for each $\varepsilon \in \mathscr{E}$ rational functions $p_1^\varepsilon, \dots, p_{d/2}^\varepsilon, q_1^\varepsilon, \dots, q_{d/2}^\varepsilon$ on $\mathfrak{g}^*$ with the following properties: For each orbit O contained in Ω_ε the restrictions $q_1^O, \dots, q_{d/2}^O$ of $q_1^\varepsilon, \dots, q_{d/2}^\varepsilon$ to O and the restrictions $p_1^O, \dots, p_{d/2}^O$ of $p_1^\varepsilon, \dots, p_{d/2}^\varepsilon$ to O define polynomial canonical coordinates on O (Theorem 5.1.1). Actually we do more than that: We even give an algorithm for constructing these rational functions $p_1^\varepsilon, \dots, p_{d/2}^\varepsilon, q_1^\varepsilon, \dots, q_{d/2}^\varepsilon$. This means that we have in particular an algorithm for explicitly constructing global canonical coordinates on each coadjoint orbit of G. In §6 we give an example of calculations carried out by means of this algorithm.

In [11] M. Vergne proved that if G is nilpotent, then there exists a Zariski open, G-invariant subset Ω of $\mathfrak{g}^*$ and rational functions $\tilde{p}_1, \dots, \tilde{p}_{d/2}, \tilde{q}_1, \dots, \tilde{q}_{d/2}$ such that these functions when restricted to any coadjoint orbit O contained in Ω define canonical coordinates on O. This result is contained as a special case (corresponding to generic orbits) in our Theorem 5.1.1. Our method of proof is entirely different from the one in [11]; in particular we proceed constructively, without induction on the dimension of the group.

1. Preliminaries

In the following we shall discuss various questions related to the fact that a connected, simply connected nilpotent Lie group can be considered as $\mathbf{R}^m$ equipped with a group product which can be expressed by polynomial functions. The main result is Proposition 1.7.4. For the convenience of the reader we repeat certain well-known facts for which there exist only scattered references.

1.1. Polynomial structures. In dealing with nilpotent Lie groups we shall find the following terminology useful.

Let M be an m-dimensional smooth manifold with atlas $\mathscr{A}_M$. We say that a subatlas $\mathscr{A}_M^0$ of $\mathscr{A}_M$ defines a polynomial structure on M if (i) all charts in $\mathscr{A}_M^0$ are defined on all of M and have image $\mathbf{R}^m$, and (ii) for any two charts σ, τ in $\mathscr{A}_M^0$ we have that $\sigma \circ \tau^{-1}: \mathbf{R}^m \to \mathbf{R}^m$ and $\tau \circ \sigma^{-1}: \mathbf{R}^m \to \mathbf{R}^m$ are both polynomial maps. So, if M carries a polynomial structure it is in particular diffeomorphic to $\mathbf{R}^m$.

If M and N are smooth manifolds of dimension m and n, respectively, equipped with polynomial structures $\mathscr{A}_M^0$ and $\mathscr{A}_N^0$, respectively, we say that a map $f: M \to N$ is polynomial if the map $\tau \circ f \circ \sigma^{-1}: \mathbf{R}^m \to \mathbf{R}^n$ is a polynomial map for one (hence all) chart(s) $\sigma \in \mathscr{A}_M^0$ and one (hence all) chart(s) $\tau \in \mathscr{A}_N^0$. We say that f is a bi-polynomial diffeomorphism if f is a diffeomorphism and if both f and f^{-1} are polynomial.

If M and N are smooth manifolds with polynomial structures, then the smooth manifold $M \times N$ carries a natural polynomial structure.

The Euclidian space $\mathbf{R}^m$ carries a natural polynomial structure. A bi-polynomial diffeomorphism $f: \mathbf{R}^m \to \mathbf{R}^m$ is a diffeomorphism such that both f and f^{-1} are polynomial maps. A finite dimensional real vector space V over $\mathbf{R}$ likewise carries a natural polynomial structure.

Let M be a smooth manifold equipped with a polynomial structure $\mathscr{A}_M^0$. A chart σ on M is said to be a *polynomial chart*, if σ is a bi-polynomial diffeomorphism from M onto $\mathbf{R}^m$. The space of all polynomial functions $f: M \to \mathbf{C}$ is denoted $\mathscr{P}(M)$. Any object defined on $\mathbf{R}^m$ which is invariant under bi-polynomial diffeomorphisms can also be defined on M. In particular we can speak of rapidly decreasing functions $\mathscr{S}(M)$ on M, tempered distributions $\mathscr{S}'(M)$, smooth functions of polynomial growth $\mathscr{O}_M(M)$, polynomial differential operators $\mathscr{A}(M)$ (of order at most k, $\mathscr{A}^k(M)$), polynomial vector fields $\mathscr{U}(M)$, polynomial differential forms etc.

1.2. The polynomial structure on a connected, simply connected nilpotent Lie group. Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. It is well known that the exponential map $\exp: \mathfrak{g} \to G$ is a diffeomorphism. We shall equip G with the polynomial structure obtained by transporting the natural polynomial structure on $\mathfrak{g}$ to G via the exponential map.

Let $\mathscr{B}: X_1, \dots, X_m$ be a basis in $\mathfrak{g}$. We shall say that the basis $\mathscr{B}$ is *of subalgebra type* if the subspace $\mathfrak{g}_j = \mathbf{R}X_1 \oplus \cdots \oplus \mathbf{R}X_j$ is a subalgebra in $\mathfrak{g}$ for each $j = 1, \dots, m$. If $\mathscr{B}$ is such a basis then actually $\mathfrak{g}_{j-1}$ is an ideal in $\mathfrak{g}_j$ for each $j = 2, \dots, m$. Recall that the basis $\mathscr{B}$ is said to be a Jordan-Hölder basis if $\mathfrak{g}_j$ is an ideal in $\mathfrak{g}$ for each $j = 1, \dots, m$.

Let $\mathscr{B}: X_1, \dots, X_m$ be a basis in $\mathfrak{g}$ of subalgebra type. It is well-known that the map $\mathbf{R}^m \to G: (x_1, \dots, x_m) \to \exp x_m X_m \cdots \exp x_1 X_1$ is a diffeomorphism. We define $\sigma_G^{\mathscr{B}}$ to be the inverse of this map, i.e.,

$$\sigma_G^{\mathscr{B}} : \exp x_m X_m \cdots \exp x_1 X_1 \to (x_1, \dots, x_m),$$

and $\sigma_G^{\mathscr{B}}$ is then a chart on G.

Lemma 1.2.1. *Write*

$$(\sigma_G^{\mathscr{B}})^{-1}(x_1, \dots, x_m) = \exp(Q_1(x)X_1 + \cdots + Q_m(x)X_m),$$
$$\sigma_G^{\mathscr{B}}(\exp(u_1 X_1 + \cdots + u_m X_m)) = (P_1(u), \dots, P_m(u)),$$

$x = (x_1, \dots, x_m) \in \mathbf{R}^m$, $u = (u_1, \dots, u_m) \in \mathbf{R}^m$. *If* $\mathscr{B}$ *is a Jordan-Hölder basis then* $Q_1, \dots, Q_m$, $P_1, \dots, P_m$ *are polynomial functions and we have*

$$Q_j(x) = x_j + \overline{Q}_j(x_{j+1}, \dots, x_m),$$
$$P_j(u) = u_j + \overline{P}_j(u_{j+1}, \dots, u_m)$$

for each $j = 1, \dots, m$, *where* $\overline{Q}_j$, $\overline{P}_j$ *are polynomial functions with properties as indicated (i.e., depending only on the variables* $x_{j+1}, \dots, x_m$, $u_{j+1}, \dots, u_m$, *respectively).*

Proof. The statement about $Q_1, \dots, Q_m$ follows immediately from the Campbell-Baker-Hausdorff formula, and the statement about $P_1, \dots, P_m$ is then clear, cf. [6].

Corollary 1.2.2. *If* $\mathscr{B}$ *is a Jordan-Hölder basis then* $\sigma_G^{\mathscr{B}}$ *is a polynomial chart.*

For $x, y \in \mathbf{R}^m$ we set $x \cdot y = \sigma_G^{\mathscr{B}}((\sigma_G^{\mathscr{B}})^{-1}(x)(\sigma_G^{\mathscr{B}})^{-1}(y))$. In other words, we identify G with $\mathbf{R}^m$ via the chart $\sigma_G^{\mathscr{B}}$, and carry the group product on G over to $\mathbf{R}^m$. With this notation we have

Lemma 1.2.3. *If* $\mathscr{B}$ *is a Jordan-Hölder basis then the function* $P(x, y) = x \cdot y$ *is a polynomial function on* $\mathbf{R}^m \times \mathbf{R}^m$, *and writing*

$$P(x, y) = (P_1(x, y), \dots, P_m(x, y))$$

we have

$$\begin{aligned} P_j(x, y) &= P_j(x_j, \dots, x_m, y_j, \dots, y_m) \\ &= x_j + y_j + \overline{P}_j(x_{j+1}, \dots, x_m, y_{j+1}, \dots, y_m), \end{aligned}$$

for $1 \le j \le m$, *where we have written* $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, *and where* $\overline{P}_j$ *are polynomial functions with properties as indicated.*

Proof. This follows immediately from the Campbell-Baker-Hausdorff formula and Lemma 1.2.1, cf. [6].

The following result will be useful later. Let e be a nonempty subset of $\{1, \dots, m\}$, and set e' to be the complement of e in $\{1, \dots, m\}$.

Lemma 1.2.4. *There exist unique functions* $S\colon \mathbf{R}^m \to \mathbf{R}^m$ *and* $K\colon \mathbf{R}^m \to \mathbf{R}^m$ *such that*

$$(*) \qquad \begin{cases} S_j = 0 & \text{for } j \in e', \\ K_j = 0 & \text{for } j \in e, \\ x = S(x)\cdot K(x). \end{cases}$$

These functions are polynomial functions, and

$$S_j(x) = x_j + \overline{S}_j(x_{j+1}, \dots, x_m) \quad \text{for } j \in e,$$
$$K_j(x) = x_j + \overline{K}_j(x_{j+1}, \dots, x_m) \quad \text{for } j \in e',$$

where $\overline{S}_j$ *and* $\overline{K}_j$ *are polynomial functions with properties as indicated.*

Proof. Suppose first that we have found functions satisfying $(*)$. We shall prove by induction on j (starting at m and ending at 1) that S_j, K_j, $j = m, \dots, 1$, are uniquely determined and that they are polynomial functions with properties as indicated. We have for each $j = m, \dots, 1$

$$\begin{aligned} x_j &= P_j(S(x), K(x)) \\ &= S_j(x) + K_j(x) + \overline{P}_j(S_{j+1}(x), \dots, S_m(x), K_{j+1}(x), \dots, K_m(x)). \end{aligned}$$

In particular $x_m = S_m(x) + K_m(x)$. Therefore, if $m \in e$ we have $S_m(x) = x_m$ and $K_m(x) = 0$; if $m \in e'$ we have $S_m(x) = 0$ and $K_m(x) = x_m$. This means that we can start the induction (at m and descending to 1). Suppose now that the claim has been proved for all values greater than j. We shall then show that it is valid also for the value j. If $j \in e$ we have $K_j(x) = 0$ and by the induction hypothesis

$$\begin{aligned} S_j(x) = x_j - \overline{P}_j(S_{j+1}(x_{j+1}, \dots, x_m), \dots, S_m(x_m), \\ K_{j+1}(x_{j+1}, \dots, x_m), \dots, K_m(x_m)) \end{aligned}$$

which shows that S_j is uniquely determined and that it is a polynomial function, since $S_m, \dots, S_{j+1}$, $K_m, \dots, K_{j+1}$ are uniquely determined and polynomial functions, and that $S_j(x) = x_j + \overline{S}(x_{j+1}, \dots, x_m)$. Similarly for K_j if $j \in e'$.

The existence of the functions S and K are proved similarly by recursion, starting with S_m, K_m and descending to S_1, K_1.

Suppose that $\mathscr{B}' : X_1', \dots, X_m'$ is another basis in $\mathfrak{g}$ and suppose that $X_j' - X_j \in \mathfrak{g}_{j-1}$ for $j = 1, \dots, m$.

Lemma 1.2.5. *Set* $T = (T_1, \dots, T_m)\colon \mathbf{R}^m \to \mathbf{R}^m$ *to be the polynomial function* $\sigma_G^{\mathscr{B}'} \circ \sigma_G^{\mathscr{B}^{-1}}$. *If* $\mathfrak{g}_0, \dots, \mathfrak{g}_m$ *is a Jordan-Hölder sequence we have*

$$T_j(x) = x_j + \overline{T}_j(x_{j+1}, \dots, x_m),$$

$x = (x_1, \dots, x_m)$, where $\overline{T}_j$ are polynomial functions with properties as indicated.

Proof. This follows immediately from Lemma 1.2.1 and Lemma 1.2.3.

For $x \in \mathbf{R}^m$ we set $x^{-1} = \sigma_G^{\mathscr{B}}(((\sigma_G^{\mathscr{B}})^{-1}(x))^{-1})$. With this notation we have

Lemma 1.2.6. *If $\mathscr{B}$ is a Jordan-Hölder basis then the function $I(x) = x^{-1}$ is a polynomial function on $\mathbf{R}^m$, and writing $I(x) = (I_1(x), \dots, I_m(x))$ we have*

$$I_j(x) = -x_j + \overline{I}_j(x_{j+1}, \dots, x_m),$$

where we have written $x = (x_1, \dots, x_m)$, and where $\overline{I}_j$ are polynomial functions with properties as indicated.

Proof. This follows immediately from Lemma 1.2.3.

Corollary 1.2.7. *The group operations $G \times G \to G$: $(s, t) \to st$ and $G \to G$: $s \to s^{-1}$ are polynomial maps.*

Suppose that $\mathfrak{g}_0$ is a subalgebra in $\mathfrak{g}$ of codimension 1, and let $X \in \mathfrak{g} \setminus \mathfrak{g}_0$. Denoting by G_0 the analytic subgroup corresponding to $\mathfrak{g}_0$ it is well known that the map $\mathbf{R} \times G_0 \to G$ given by $(t, s) \to \exp tXs$ is a diffeomorphism. Actually we have

Lemma 1.2.8. *The map*

$$\mathbf{R} \times G_0 \to G : (t, s) \to \exp tXs$$

is a bi-polynomial diffeomorphism.

Proof. Let σ be some polynomial chart on G and let τ be some polynomial chart on G_0. We have to show that the map $\varphi \colon \mathbf{R}^m \to \mathbf{R}^m$ given by $\varphi : (t, x_1, \dots, x_{m-1}) \to \sigma(\exp tX\tau^{-1}(x_1, \dots, x_{m-1})$ is a bi-polynomial diffeomorphism. Now since we know that actually $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$ we can find a Jordan-Hölder basis $\mathscr{B} : X_1, \dots, X_m$ in $\mathfrak{g}$ such that $\mathscr{B}_0 : X_1, \dots, X_{m-1}$ is a Jordan-Hölder basis in $\mathfrak{g}_0$ and such that $X_m = X$. Using $\sigma = \sigma_G^{\mathscr{B}}$ and $\tau = \sigma_{G_0}^{\mathscr{B}_0}$ we get that $\varphi(t, x_1, \dots, x_{m-1}) = (x_1, \dots, x_{m-1}, t)$, so φ is polynomial. This ends the proof of the lemma.

Lemma 1.2.9. *The chart $\sigma_G^{\mathscr{B}}$ is a polynomial chart for any basis $\mathscr{B}$ of subalgebra type.*

Proof. The proof is by induction on the dimension of G. The case $\dim G = 1$ being trivial, assume the result has been proved for all dimensions less that $m = \dim G$. Let $\mathscr{B} \colon X_1, \dots, X_m$ be the given basis of subalgebra type, and set $\mathfrak{g}_0 = \mathbf{R}X_1 \oplus \cdots \oplus \mathbf{R}X_{m-1}$. Then $\mathfrak{g}_0$ is a subalgebra of codimension 1 in $\mathfrak{g}$. Moreover, $\mathscr{B}_0 \colon X_1, \dots, X_{m-1}$ is a basis in $\mathfrak{g}_0$ of subalgebra type, and $X_m \notin \mathfrak{g}_0$. It follows from the induction hypothesis that $\sigma_{G_0}^{\mathscr{B}_0}$ is a polynomial chart on $G_0 = \exp \mathfrak{g}_0$, and since

$$(\sigma_G^{\mathscr{B}})^{-1}(x_1, \dots, x_m) = \exp x_m X_m (\sigma_{G_0}^{\mathscr{B}_0})^{-1}(x_1, \dots, x_{m-1})$$

it follows from Lemma 1.2.8 that $\sigma_G^{\mathscr{B}}$ is a polynomial chart.

1.3. The polynomial structure on a simply connected homogeneous space of a nilpotent Lie group. Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and let H be the analytic subgroup corresponding to $\mathfrak{h}$. We shall equip G/H with a polynomial structure.

First some notation. Let $Y_1, \ldots, Y_n$ be elements in $\mathfrak{g}$. We say that $Y_1, \ldots, Y_n$ is a basis in $\mathfrak{g}$ (mod $\mathfrak{h}$) if $Y_1, \ldots, Y_n$ is a linearly independent set and if $\mathfrak{g} = \mathbf{R}Y_1 \oplus \cdots \oplus \mathbf{R}Y_n \oplus \mathfrak{h}$. We say that $Y_1, \ldots, Y_n$ is a basis in $\mathfrak{g}$ (mod $\mathfrak{h}$) of subalgebra type if each of the subspaces $\mathfrak{h}_j = \mathbf{R}Y_1 \oplus \cdots \oplus \mathbf{R}Y_j \oplus \mathfrak{h}$ is a subalgebra in $\mathfrak{g}$. If this is so, $\mathfrak{h}_{j-1}$ is actually an ideal in $\mathfrak{h}_j$, $j = 0, 1, \ldots, n$ (where $\mathfrak{h}_0 = \mathfrak{h}$).

Let $Y_1, \ldots, Y_n$ be a basis in $\mathfrak{g}$ (mod $\mathfrak{h}$) of subalgebra type. Then it is well-known that the map $\sigma : G/H \to \mathbf{R}^n$ given by

$$\sigma : \exp t_n Y_n \cdots \exp t_1 Y_1 H \to (t_1, \ldots, t_n)$$

is a diffeomorphism. Suppose now that $Z_1, \ldots, Z_n$ is another basis in $\mathfrak{g}$ (mod $\mathfrak{h}$) of subalgebra type, and define analogously the map $\tau : G/H \to \mathbf{R}^n$ by

$$\tau : \exp t_n Z_n \cdots \exp t_1 Z_1 H \to (t_1, \ldots, t_n).$$

Then we have

Lemma 1.3.1. *The maps $\sigma \circ \tau^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ and $\tau \circ \sigma^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ are polynomial.*

Proof. Let $W_1, \ldots, W_{m-n}$ be a basis in $\mathfrak{h}$ of subalgebra type. Then $W_1, \ldots, W_{m-n}, Y_1, \ldots, Y_n$ and $W_1, \ldots, W_{m-n}, Z_1, \ldots, Z_n$ are both bases of subalgebra type in $\mathfrak{g}$, and the result now follows from Lemma 1.2.9.

Lemma 1.3.1 allows us to define a polynomial structure on G/H consisting of all possible charts σ arising as above.

Lemma 1.3.2. *A function $\varphi : G/H \to \mathbf{C}$ is a polynomial function if and only if the function $\tilde{\varphi} : G \to \mathbf{C}$ given by $\tilde{\varphi}(s) = \varphi(sH)$ is a polynomial function on G.*

Proof. Let $Y_1, \ldots, Y_n$ be a basis in $\mathfrak{h}$ of subalgebra type, and let $Y_{n+1}, \ldots, Y_m$ be a basis in $\mathfrak{g}$ (mod $\mathfrak{h}$) of subalgebra type. Then $Y_1, \ldots, Y_m$ is a basis in $\mathfrak{g}$ of subalgebra type. Let $\tilde{\sigma}$ be the chart on G/H associated with the basis $Y_1, \ldots, Y_m$, and let σ be the chart on G/H associated with the basis $Y_{n+1}, \ldots, Y_m$ (mod $\mathfrak{h}$) (see §1.3). We then have

$$\begin{aligned}
\tilde{\varphi} \circ \tilde{\sigma}^{-1}(t_1, \ldots, t_m) &= \tilde{\varphi}(\exp t_m Y_m \cdots \exp t_1 Y_1) \\
&= \varphi(\exp t_m Y_m \cdots \exp t_1 Y_1 H) \\
&= \varphi(\exp t_m Y_m \cdots \exp t_{n+1} Y_{n+1} H) \\
&= \varphi \circ \sigma^{-1}(t_{n+1}, \ldots, t_m).
\end{aligned}$$

This shows that $\tilde{\varphi} \circ \tilde{\sigma}^{-1}$ is a polynomial function if and only if $\varphi \circ \sigma^{-1}$ is a polynomial function, and this ends the proof of the lemma.

1.4. Action of nilpotent Lie groups on a simply connected homogeneous spaces. Let $\mathscr{B}: X_1, \dots, X_m$ be a Jordan-Hölder basis in $\mathfrak{g}$, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and set $H = \exp \mathfrak{h}$. Define the set $e = \{1 \le j \le m \mid X_j \notin \mathfrak{g}_{j-1} + \mathfrak{h}\}$, and write $e = \{j_1 < \dots < j_d\}$. Then $X_{j_1}, \dots, X_{j_d}$ is a basis for $\mathfrak{g} \pmod{\mathfrak{h}}$ of subalgebra type. In fact, set $\mathfrak{h}_0 = \mathfrak{h}$, and set $\mathfrak{h}_r = \mathbf{R}X_{j_r} \oplus \dots \oplus \mathbf{R}X_{j_1} \oplus \mathfrak{h}$ for $1 \le r \le d$. Clearly we have $\mathfrak{g}_j + \mathfrak{h} = \mathfrak{h}_r$ for $j_r \le j < j_{r+1}$. But then $[\mathfrak{h}_r, \mathfrak{h}_{r-1}] = [\mathfrak{g}_{j_r} + \mathfrak{h}, \mathfrak{g}_{j_{r-1}} + \mathfrak{h}] \subset [\mathfrak{g}, \mathfrak{g}_{j_r}] + [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{g}_{j_r - 1} + \mathfrak{h} = \mathfrak{h}_{r-1}$, and this shows the claim. It follows that the map $\sigma^{\mathscr{B}}_{G/H} : G/H \to \mathbf{R}^d$ given by

$$\sigma^{\mathscr{B}}_{G/H} : \exp t_d X_{j_d} \cdots \exp t_1 X_{j_1} H \to (t_1, \dots, t_d)$$

is a polynomial chart. We shall call $\sigma^{\mathscr{B}}_{G/H}$ for the $\mathscr{B}$-chart of G/H.

For $x \in \mathbf{R}^m$ and $u \in \mathbf{R}^d$ we write $x \cdot u = \sigma^{\mathscr{B}}_{G/H}((\sigma^{\mathscr{B}}_G)^{-1}(x)(\sigma^{\mathscr{B}}_{G/H})^{-1}(u))$. In other words, we identify G with $\mathbf{R}^m$ via $\sigma^{\mathscr{B}}_G$ and G/H with $\mathbf{R}^d$ via $\sigma^{\mathscr{B}}_{G/H}$ and transport the action of G on G/H to a smooth action of $\mathbf{R}^m$ on $\mathbf{R}^d$. With this notation we have

Lemma 1.4.1. *The function $Q(x, u) = x \cdot u$ is a polynomial function on $\mathbf{R}^m \times \mathbf{R}^d$, and writing $Q(x, u) = (Q_1(x, u), \dots, Q_d(x, u))$ we have*

$$Q_r(x, u) = x_{j_r} + u_r + \overline{Q}_r(x_{j_r+1}, \dots, x_m, u_{r+1}, \dots, u_d),$$

for $1 \le r \le d$, where we have written $x = (x_1, \dots, x_m)$, $u = (u_1, \dots, u_d)$, and where $\overline{Q}_r$ are polynomial functions with properties as indicated.

Proof. If $j \notin e$ there exists $X'_j \in \mathfrak{h}$ such that $X_j - X'_j \in \mathfrak{g}_{j-1}$. Using Lemma 1.2.5 we see that it is no loss of generality to assume that $X_j \in \mathfrak{h}$ for all $j \notin e$. For $u \in \mathbf{R}^d$ we then set $y_{j_r} = u_r$ for $r = 1, \dots, d$ and $y_j = 0$ for $j \notin e$, and we set $z = x \cdot y$. Then with the notation from Lemma 1.2.4 we have that $(x \cdot u)_r = S_{j_r}(z)$, i.e., $Q_r(x, u) = S_{j_r}(z) = x_{j_r} + y_{j_r} + \overline{S}_{j_r}(x_{j_r+1}, \dots, x_m, y_{j_r+1}, \dots, y_m) = x_{j_r} + u_r + \overline{Q}_r(x_{j_r+1}, \dots, x_m, u_{r+1}, \dots, u_d)$. This ends the proof of the lemma.

Corollary 1.4.2. *The map $(s, a) \to sa: \ G \times G/H \to G/H$ is a polynomial map.*

1.5. Polynomial vector fields on simply connected homogeneous spaces of nilpotent Lie groups. Let $\mathfrak{h}$ be a subalgebra in $\mathfrak{g}$, and set $H = \exp \mathfrak{h}$. For $X \in \mathfrak{g}$ we define the vector field $\tilde{X}$ on G/H as follows:

$$\tilde{X} f(a) = \frac{d}{dv} f(\exp -vXa)\bigg|_{v=0},$$

$a \in G/H$, $f \in \mathscr{E}(G/H)$. It follows from Corollary 1.4.2 that $\tilde{X}$ is a polynomial vector field on G/H.

Let now $\mathscr{B}: X_1, \dots, X_m$ be a Jordan-Hölder basis in $\mathfrak{g}$, set $e = \{1 \le j \le m \mid X_j \notin \mathfrak{g}_{j-1} + \mathfrak{h}\}$, and write $e = \{k_1 < \dots < k_{d/2}\}$.

Lemma 1.5.1. *The space of polynomial vector fields* $\mathscr{U}(G/H)$ *on* G/H *is a free* $\mathscr{P}(G/H)$*-module with basis* $\tilde{X}_{k_1}, \ldots, \tilde{X}_{k_{d/2}}$.

Proof. We consider the $\mathscr{B}$-chart $\sigma_{G/H}^{\mathscr{B}}$ of G/H given by

$$\sigma_{G/H}^{\mathscr{B}} : \exp t_{d/2} X_{k_{d/2}} \cdots \exp t_1 X_{k_1} \to (t_1, \ldots, t_{d/2}),$$

see §1.4. Using this chart we identify G/H with $\mathbf{R}^{d/2}$ and we can think of $\tilde{X}_j$ as polynomial vector fields on $\mathbf{R}^{d/2}$. Similarly we identify G with $\mathbf{R}^m$ via the $\mathscr{B}$-chart $\sigma_G^{\mathscr{B}}$, see §1.1. Let us then compute $\tilde{X}_{k_s}$, $1 \leq s \leq d/2$. By Lemma 1.4.1 the action of $G = \mathbf{R}^m$ on $G/H = \mathbf{R}^{d/2}$ is given by $x \cdot u = Q(x, u)$, $x \in \mathbf{R}^m$, $u \in \mathbf{R}^{d/2}$, where $Q = (Q_1, \ldots, Q_{d/2})$ is a polynomial function such that $Q_r(x, u) = x_{k_r} + u_r + \overline{Q}_r(x_{k_r+1}, \ldots, x_m, u_{r+1}, \ldots, u_{d/2})$. Setting

$$a_{rs}(u) = \frac{d}{dv} Q_s(\exp -vX_{k_r} \cdot u)|_{v=0}$$

for $1 \leq r, s \leq d/2$, $u \in \mathbf{R}^{d/2}$, we get, using the chain rule:

$$\begin{aligned} \tilde{X}_{k_r} f(u) &= \frac{d}{dv} f(\exp -vX_{k_r} \cdot u)\Big|_{v=0} \\ &= \frac{d}{dv} f(Q(\exp -vX_{k_r}, u))\Big|_{v=0} = \sum_{s=1}^{d/2} \frac{\partial f}{\partial u_s}(u) a_{rs}(u). \end{aligned}$$

Now

$$Q_s(\exp -vX_{k_r}, u) = \begin{cases} u_s & \text{for } s > r, \\ u_s - v + \overline{Q}_s(\exp -vX_{k_r}, u_{s+1}, \ldots, u_m) & \text{for } r = s, \\ u_s + \overline{Q}_s(\exp -vX_{k_r}, u_{s+1}, \ldots, u_m) & \text{for } r < s. \end{cases}$$

This implies that

$$a_{rs}(u) = \begin{cases} 0 & \text{for } s > r, \\ -1 & \text{for } s = r, \\ a_{rs}(u_{s+1}, \ldots, u_n) & \text{for } s < r. \end{cases}$$

It follows that the martix $\mathbf{A}(u) = [a_{rs}]_{1 \leq r, s \leq d/2}$ is a lower triangular matrix with polynomial entries and with -1 in the diagonal. But then also the inverse matrix $\mathbf{B}(u) = \mathbf{A}(u)^{-1}$ is of the same kind. Now writing $\mathbf{B}(u) = [b_{rs}]_{1 \leq r, s \leq d/2}$ we have

$$\frac{\partial}{\partial u_r} = \sum_{s=1}^{d/2} b_{rs}(u) \tilde{X}_{k_s}.$$

It follows that $\tilde{X}_{k_1}, \ldots, \tilde{X}_{k_{d/2}}$ generates $\mathscr{U}(G/H)$ as a $\mathscr{P}(G/H)$-module. That $\tilde{X}_{k_1}, \ldots, \tilde{X}_{k_{d/2}}$ is actually a basis for $\mathscr{U}(G/H)$ follows immediately from the fact that the matrix $\mathbf{A}(u)$ is regular. This ends the proof of the lemma.

For $s \in G$ define the endomorphism $\gamma_{G/H}(s)$ of $\mathscr{E}(G/H)$ by $\gamma_{G/H}(s)f(a) = f(s^{-1}a)$, $a \in G/H$. Then $s \to \gamma_{G/H}(s)$ is a differentiable representation of G. For $X \in \mathfrak{g}$ and $f \in \mathscr{E}(G/H)$ we have $d\gamma_{G/H}(X)f = \tilde{X}f$. Let I denote the identity operator.

Lemma 1.5.2. *The space $\mathscr{A}^1(G/H)$ is a free $\mathscr{P}(G/H)$-module with basis I, $d\gamma_{G/H}(X_{k_1}), \dots, d\gamma_{G/H}(X_{k_{d/2}})$.*

Proof. Let $D \in \mathscr{A}^1(G/H)$. Since there exists a bi-polynomial diffeomorphism from G/H to $\mathbf{R}^d$, D can be written uniquely as $Df = \xi(f) + \varphi_0 f$, where ξ is a polynomial vector field on G/H and $\varphi_0 \in \mathscr{P}(G/H)$. The lemma now follows from Lemma 1.5.1.

1.6. The polynomial structure on coadjoint orbits of nilpotent Lie groups. Let G, $\mathfrak{g}$ be as before, and let $O = Gg$ be the coadjoint orbit through $g \in \mathfrak{g}^*$. The orbit O is equipped with the smooth structure it gets from the identification $G/G_g \simeq O$. We also carry over the polynomial structure already defined on G/G_g to O. It is easily seen that this definition of a polynomial structure on O is independent of the choice of $g \in O$ (Corollary 1.4.2).

Given a Jordan-Hölder basis $\mathscr{B} : X_1, \dots, X_m$ in $\mathfrak{g}$ there exist indices $j_1 < \cdots < j_d$ (the "jump indices"of O relative to $\mathscr{B}$) and polynomial functions $R_1, \dots, R_m$ in the variables $(y_{j_1}, \dots, y_{j_d})$ such that $R_{j_u}(y) = y_{j_u}$, $1 \le u \le d$, and such that O can be described as follows

$$O = \left\{ l = \sum_{j=1}^{m} R_j(y_{j_1}, \dots, y_{j_d}) l_j \mid (y_{j_1}, \dots, y_{j_d}) \in \mathbf{R}^d \right\},$$

where $l_1, \dots, l_m$ is the basis in $\mathfrak{g}^*$ dual to $X_1, \dots, X_m$. The set of jump indices J_g associated with $g \in \mathfrak{g}^*$ can be found as

$$J_g = \{1 \le j \le m \mid \mathfrak{g}_j \not\subset \mathfrak{g}_{j-1} + \mathfrak{g}_g\},$$

cf. §4.2. Define the map $\sigma_O^{\mathscr{B}} : O \to \mathbf{R}^d$ by

$$\sigma_O^{\mathscr{B}} : l = \sum_{j=1}^{m} R_j(y_{j_1}, \dots, y_{j_d}) l_j \to (y_{j_1}, \dots, y_{j_d}) = (\langle l, X_{j_1}\rangle, \dots, \langle l, X_{j_d}\rangle).$$

Lemma 1.6.1. *The map $\sigma_O^{\mathscr{B}}$ is a polynomial chart on O.*

Proof. Let $g \in O$. It follows from [10, p. 265] that there exists a basis $Y_d, \dots, Y_1$ of $\mathfrak{g}$ (mod $\mathfrak{g}_g$) of subalgebra type such that if we write

$$\exp t_1 Y_1 \cdots \exp t_d Y_d g = \sum_{j=1}^{m} Q_j(t) l_j,$$

where $t = (t_1, \dots, t_d)$, then the functions $Q_j(t)$ are polynomial functions with the following properties: For $j_r \le j < j_{r+1}$ we have

$$Q_j(t) = Q_j(t_1, \dots, t_r, 0, \dots, 0)$$

and

$$Q_{j_r}(t) = t_r + Q_{j_r}(t_1, \dots, t_{r-1}, 0, \dots, 0).$$

Let now σ be the polynomial chart associated with $Y_d, \dots, Y_1$. For $l \in O$, write $\sigma(l) = t = (t_1, \dots, t_d)$ and $\sigma_O^{\mathscr{B}}(l) = y = (y_1, \dots, y_d)$. Then

$$\begin{aligned}
y_1 &= t_1 + Q_{j_1}(0, \dots, 0) \\
y_2 &= t_2 + Q_{j_2}(t_1, 0, \dots, 0) \\
y_3 &= t_3 + Q_{j_3}(t_1, t_2, 0, \dots, 0) \\
&\vdots \\
y_d &= t_d + Q_{j_d}(t_1, \dots, t_{d-1}, 0).
\end{aligned}$$

From this it follows immediately that $\sigma_O^{\mathscr{B}}$ is a polynomial chart.

Corollary 1.6.2. *A function $f: O \to \mathbf{C}$ is polynomial if and only if it is the restriction to O of a polynomial function on $\mathfrak{g}^*$.*

Proof. If $\varphi \in \mathscr{P}(O)$, then $f = \varphi \circ (\sigma_O^{\mathscr{B}})^{-1}$ is a polynomial function on $\mathbf{R}^d$, and φ is the restriction to O of the function $l \to f(\langle l, X_{j_1}\rangle, \dots, \langle l, X_{j_d}\rangle)$ on $\mathfrak{g}^*$. If $\varphi \in \mathscr{P}(\mathfrak{g}^*)$ we can write $\varphi = f(\langle X_1, l\rangle, \dots, \langle X_m, l\rangle)$, where f is a polynomial function on $\mathbf{R}^m$. But then $\varphi \circ (\sigma_O^{\mathscr{B}})^{-1}(y) = f(R_1(y), \dots, R_m(y))$. Since $R_1, \dots, R_m$ are polynomial functions it follows that the restriction of φ to O is a polynomial function.

1.7. Rapidly decreasing sections and polynomial coefficient differential operators in homogeneous line bundles of nilpotent Lie groups. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and let $H = \exp \mathfrak{h}$. Let $Y_1, \dots, Y_n$ be a basis of $\mathfrak{g}$ (mod $\mathfrak{h}$) of subalgebra type, and let $\sigma : G/H \to \mathbf{R}^n$ given by

$$\sigma : \exp t_n Y_n \cdots \exp t_1 Y_1 H \to (t_1, \dots, t_n)$$

be the corresponding chart on G/H. We define $c : G \to G/H$ to be the quotient map and $\kappa : G/H \to G$ to be the map

$$\kappa : \exp t_n Y_n \cdots \exp t_1 Y_1 H \to \exp t_n Y_n \cdots \exp t_1 Y_1.$$

In this way κ is a cross-section for c: $c \circ \kappa = \mathrm{id}_{G/H}$. We also define the map $\theta : \mathbf{R}^n \to G$ by

$$\theta : (t_1, \dots, t_n) \to \exp t_n Y_n \cdots \exp t_1 Y_1.$$

The maps c, κ and θ are polynomial maps, and we have

$$\theta \circ \sigma = \kappa.$$

Define the map $a : G \to H$ by $s = \kappa(c(s))a(s)$ for all $s \in G$, i.e., $a(s) = (\kappa(c(s)))^{-1}s$. Then a is a polynomial map, and $a(sh) = a(s)h$ for all $s \in G$, $h \in H$. We also define the map $A : G \to \mathfrak{h}$ by $A(s) = \log(a(s))$. Clearly A is also a polynomial map.

Suppose now that $\chi : H \to \mathbf{T}$ is a unitary character, and let $g \in \mathfrak{g}^*$ be such that $\chi(\exp X) = e^{i\langle g, X\rangle}$ for all $X \in \mathfrak{h}$. In [9] we have defined $\mathscr{E}(G, \chi)$ to be the set of all smooth functions f on G such that $f(sh) = \chi(h)^{-1} f(s)$ for all $s \in G$, $h \in H$. We define a linear bijection $V : \mathscr{E}(G, \chi) \to \mathscr{E}(\mathbf{R}^n)$ by

$$(Vf)(t) = f(\theta(t)), \qquad f \in \mathscr{E}(G, \chi), \ t \in \mathbf{R}^n.$$

The inverse of V is given by

$$V^{-1} h(s) = e^{i\langle g, A(s)\rangle} h(\sigma(c(s))), \qquad h \in \mathscr{E}(\mathbf{R}^n), s \in G.$$

Denote by $\mathscr{S}(\mathbf{R}^n)$ the space of rapidly decreasing functions on $\mathbf{R}^n$ (the Schwartz functions) and let $\mathscr{A}^k(\mathbf{R}^n)$ denote the space of differential operators on $\mathbf{R}^n$ of order at most k and having polynomial coefficients, and set $\mathscr{A}(\mathbf{R}^n) = \bigcup_{k=0}^{\infty} \mathscr{A}^k(\mathbf{R}^n)$ (the Weyl algebra).

Definition 1.7.1. *We define $\mathscr{S}(G, \chi)$ to be the inverse image by V of $\mathscr{S}(\mathbf{R}^n)$, and we define $\mathscr{A}(G, \chi)$ $(\mathscr{A}^k(G, \chi))$ to be the transport of $\mathscr{A}(\mathbf{R}^n)$ $(\mathscr{A}^k(\mathbf{R}^n)$, $k = 0, 1, \ldots)$ by V^{-1}.*

Lemma 1.7.2. *The definition of $\mathscr{S}(G, \chi)$ and $\mathscr{A}^k(G, \chi)$ is independent of the choice of basis of $\mathfrak{g}$ $(\operatorname{mod} \mathfrak{h})$ of subalgebra type.*

Proof. Suppose that $Y_1', \ldots, Y_n'$ is another basis of $\mathfrak{g}$ $(\operatorname{mod} \mathfrak{h})$ of subalgebra type. The objects defined relative to this basis are all marked with a prime. We define the linear bijection $W : \mathscr{E}(\mathbf{R}^n) \to \mathscr{E}(\mathbf{R}^n)$ by $W = V'V^{-1}$. We have to show that W maps $\mathscr{S}(\mathbf{R}^n)$ onto $\mathscr{S}(\mathbf{R}^n)$, and that W transports $\mathscr{A}^k(\mathbf{R}^n)$ onto $\mathscr{A}^k(\mathbf{R}^n)$. We calculate W:

$$(Wh)(t) = e^{-i\langle g, A(\theta'(t))\rangle} h(\sigma \circ \sigma'^{-1}(t)), \qquad h \in \mathscr{E}(\mathbf{R}^n), \ t \in \mathbf{R}^n.$$

Set $\varphi = \sigma \circ \sigma'^{-1}$, and set $S(t) = -\langle g, A(\theta'(t))\rangle$, $t \in \mathbf{R}^n$. Then φ is a bi-polynomial diffeomorphism,and S is a real-valued polynomial map, and we have with this notation:

$$(Wh)(t) = e^{iS(t)} h(\varphi(t)), \qquad h \in \mathscr{E}(\mathbf{R}^n), \quad t \in \mathbf{R}^n.$$

Similarly, if we define $\psi = \varphi^{-1} = \sigma' \circ \sigma^{-1}$, and $T(t) = -\langle g, A'(\theta(t))\rangle$ we have

$$(W^{-1}h)(t) = e^{iT(t)} h(\psi(t)), \qquad h \in \mathscr{E}(\mathbf{R}^n), \ t \in \mathbf{R}^n.$$

If p is a polynomial function on $\mathbf{R}^n$ we let M_p denote the multiplication operator on $\mathscr{E}(\mathbf{R}^n)$ defined by p. One easily calculates

$$W M_p W^{-1} = M_{p \circ \varphi}.$$

Moreover, if D_j denotes the differential operator $\partial/\partial t_j$ for $1 \le j \le n$, one easily calculates that

$$W D_j W^{-1} = \sum_{k=1}^{n} M_{\partial \varphi_k/\partial t_j \circ \varphi} D_k + M_{\partial T/\partial t_j \circ \varphi}.$$

From this we see immediately that W transports kth order differential operators with polynomial coefficients on $\mathbf{R}^n$ onto itself, and from this it follows that W carries $\mathscr{S}(\mathbf{R}^n)$ onto itself. This ends the proof of the lemma.

For $s \in G$ we define the endomorphism $\gamma_\chi(s)$ of $\mathscr{E}(G,\chi)$ by $\gamma_\chi(s)f(t) = f(s^{-1}t)$, $t \in G$. The map $s \to \gamma_\chi(s)$ is a differentiable representation of G.

Lemma 1.7.3. *We have that* $d\gamma_\chi(X) \in \mathscr{A}^1(G,\chi)$ *for each* $X \in \mathfrak{g}$.

Proof. We define a linear isomorphism $T\colon \mathscr{E}(G,\chi) \to \mathscr{E}(G/H)$ as follows:

$$(Tf)(a) = f(\kappa(a)), \qquad f \in \mathscr{E}(G,\chi),\ a \in G/H.$$

The inverse is given by

$$(T^{-1}h)(s) = e^{-i\langle g, A(s)\rangle} h(c(s)), \qquad h \in \mathscr{E}(G/H).$$

Clearly T is a linear isomorphism. When we transport $\mathscr{A}^1(G,\chi)$ to $\mathscr{E}(G/H)$ by T we get $\mathscr{A}^1(G/H)$ by Definition 1.7.1. We then calculate $Td\gamma_\chi(X)T^{-1}$. We get for $h \in \mathscr{E}(G/H)$, $x \in G/H$,

$$\begin{aligned} Td\gamma_\chi(X)T^{-1}h(x) = d\gamma_\chi(X)T^{-1}h(\kappa(x)) &= \frac{d}{dv}T^{-1}h(\exp -vX\kappa(x)))\Big|_{v=0} \\ &= \frac{d}{dv}e^{-i\langle g, A(\exp -vX\kappa(x))\rangle}h(\exp -vXx)\Big|_{v=0} \\ &= \frac{d}{dv}e^{-i\langle g, A(\exp -vX\kappa(x))\rangle}\Big|_{v=0} h(x) + d\gamma_{G/H}(X)h(x). \end{aligned}$$

Therefore, if we set

$$\alpha_X(a) = \frac{d}{dv}e^{-i\langle g, A(\exp -vX\kappa(x))\rangle}\Big|_{v=0} = -i\,\frac{d}{dv}\langle g, A(\exp -vX\kappa(x))\rangle\Big|_{v=0}$$

we have that $\alpha_X \in \mathscr{P}(G/H)$, and

$$Td\gamma_\chi(X)T^{-1}h = d\gamma_{G/H}(X)h + \alpha_X h$$

for $h \in \mathscr{E}(G/H)$. This proves the lemma, since $d\gamma_{G/H}(X) \in \mathscr{A}^1(G/H)$, see §1.5.

Proposition 1.7.4. *The space* $\mathscr{A}^1(G,\chi)$ *is a free* $\mathscr{P}(G/H)$*-module with basis* $I, d\gamma_\chi(X_{k_1}), \ldots, d\gamma_\chi(X_{k_{d/2}})$.

Proof. The lemma says that each element $D \in \mathscr{A}^1(G,\chi)$ can be written uniquely

$$D = \sum_{u=1}^{d/2} \varphi_u d\gamma_\chi(X_{k_u}) + \varphi_0,$$

where $\varphi_0, \varphi_1, \ldots, \varphi_{d/2}$ are in $\mathscr{P}(G/H)$. Transforming this relation by the operator T from the previous lemma we get

$$TDT^{-1} = \sum_{u=1}^{d/2} \varphi_u d\gamma_{G/H}(X_{k_u}) + \varphi_0',$$

where $\varphi_0' = \sum_{u=1}^{d/2} \varphi_u \alpha_{X_{k_u}} + \varphi_0$. The result now follows from Lemma 1.5.2.

1.8. Realizing representations. Let $g \in \mathfrak{g}^*$ and let $\mathfrak{h}$ be a real polarization at g. Set $H = \exp \mathfrak{h}$, and let χ be the unitary character of H given by $\chi(\exp X) = e^{i\langle g, X\rangle}$, $X \in \mathfrak{h}$. The representation $\pi_\chi = \operatorname{ind}_{H\uparrow G}\chi$ of G is the irreducible representation associated with the orbit $O = Gg$. The representation space of π_χ is $L^2(G,\chi)$. Let now $X_1, \dots, X_{d/2}$ be a basis for $\mathfrak{g}$ (mod $\mathfrak{h}$) of subalgebra type. Using the notation from the previous section we can define a unitary map $V\colon L^2(G,\chi) \to L^2(\mathbf{R}^{d/2})$ using the same formula for V as before. Let us then define the representation π_χ^V of G on $L^2(\mathbf{R}^{d/2})$ by

$$\pi_\chi^V(s) = V\pi_\chi(s)V^{-1}.$$

Let $b\colon G \times \mathbf{R}^{d/2} \to H$ be the polynomial function defined by $b(s,t) = a(s^{-1}\theta(t))$, $s \in G$, $t \in \mathbf{R}^{d/2}$, and define the polynomial function $B : G\times\mathbf{R}^{d/2} \to \mathfrak{h}$ by $\exp B(s,t) = b(s,t)$, $s \in G$, $t \in \mathbf{R}^{d/2}$. Further, define a polynomial action of G in $\mathbf{R}^{d/2}$ by transport of structure (i.e., $st = \sigma(s\sigma^{-1}(t))$). The representation π_χ^V is equivalent to π_χ, and we have

$$\pi_\chi^V(s)h(t) = \chi(b(s,t))^{-1}h(s^{-1}t) = e^{-i\langle g, B(s,t)\rangle}h(s^{-1}t),$$

$h \in L^2(\mathbf{R}^{d/2})$, $s \in G$, $t \in \mathbf{R}^{d/2}$.

2. Orbits, Poisson structure and quantization of nilpotent Lie groups

In a recent paper [9] I have obtained certain results about the canonical symplectic structure of connected Lie groups, and in particular of exponential solvable Lie groups. In this section we shall briefly specialize to the nilpotent case, where we obtain more detailed results.

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, and let O be a coadjoint orbit of G.

2.1. Polynomial functions and Poisson bracket. First we consider how polynomial functions on O behave under Poisson bracket.

We select a Jordan-Hölder sequence

$$\mathscr{F} : \mathfrak{g} = \mathfrak{g}_m \supset \mathfrak{g}_{m-1} \supset \cdots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0 = \{0\}$$

and a compatible Jordan-Hölder basis $\mathscr{B}\colon X_1, \dots, X_m$, i.e. $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$, $j = 1, \dots, m$.

We set

$$e = \{1 \le j \le m \mid X_j \notin \mathfrak{g}_{j-1} + \mathfrak{g}_g\}$$

for some fixed element $g \in O$, and write $e = \{j_1 < \cdots < j_d\}$, where d is the dimension of the orbit. Let $\sigma_O^{\mathscr{B}}$ be the $\mathscr{B}$-chart of O, i.e., the chart defined by $\sigma_O^{\mathscr{B}}(l) = (\langle l, X_{j_1}\rangle, \dots, \langle l, X_{j_d}\rangle)$, see §1.6. We then know that the functions $R_j = \psi^{X_j} \circ (\sigma_O^{\mathscr{B}})^{-1}$ are polynomial functions on $\mathbf{R}^d$ with $R_{j_r}(y) = y_r$,

$r = 1, \ldots, d$, where $y = (y_1, \ldots, y_d)$. Writing $\sigma_O^{\mathscr{B}} = \sigma = (y_1, \ldots, y_d)$ we have that the matrix $\mathbf{M}_\sigma(l) = [\{y_u, y_v\}(l)]_{1\le u,v\le d}$ is equal to the matrix $\mathbf{M}_e(l) = \left[\langle l, [X_{j_u}, X_{j_v}]\rangle\right]_{1\le u,v\le d}$ [9, Lemma 1.1.8]. Let now c_{ijk} be the structure constants of $\mathfrak{g}$ associated with the basis $\mathscr{B}$, i.e., $[X_i, X_j] = \sum_{k=1}^m c_{ijk} X_k$. The entries of the matrix $\mathbf{M}_\sigma(l)$ have the form $\sum_{k=1}^m c_{j_u j_v k} \psi^{X_k}(l)$, and therefore the entries $a_{uv}(y)$ of the matrix $\mathbf{M}_\sigma(\sigma^{-1}(y))$ are of the form

$$a_{uv}(y) = \sum_{k=1}^{m} c_{j_u j_v k} R_k(y);$$

in particular these entries are polynomial functions. We can now write down the Hamiltonian vector fields and the Poisson bracket in these coordinates:

$$\xi_\varphi = \sum_{u=1}^{d} \left(\sum_{v=1}^{d} a_{uv} \frac{\partial \varphi}{\partial y_v} \right) \frac{\partial}{\partial y_u},$$

$$\{\varphi, \psi\} = \sum_{u,v=1}^{d} a_{uv} \frac{\partial \varphi}{\partial y_u} \frac{\partial \psi}{\partial y_v}$$

for $\varphi, \psi \in \mathscr{E}(O)$. From these remarks we derive the following lemma:

Lemma 2.1.1. *For each $\varphi \in \mathscr{P}(O)$ the vector field ξ_φ is polynomial. Moreover, the space $\mathscr{P}(O)$ is closed under Poisson bracket.*

Let us also note that with the notation from [9, §1.1] we have that the matrix $\mathbf{\Omega}_\sigma(l) = \left[\omega_l((\partial/\partial y_u)_l, (\partial/\partial y_v)_l)\right]_{1\le u,v\le d}$ is equal to $-\mathbf{M}_\sigma^{-1}(l)$. Since the determinant $Q_e(l)$ of $\mathbf{M}_e(l)$ is constant on the orbit, [7, Lemma 1.1.1, p. 427] (see also §4.2), we see that also $\mathbf{\Omega}_\sigma(\sigma^{-1}(y))$ has polynomial entries. We therefore get

Lemma 2.1.2. *The canonical symplectic structure ω_O is a polynomial 2-form.*

2.2. Polynomial quantizable functions. Next we shall consider the structure of the space of polynomial quantizable functions associated with a given real polarization. The main result is Proposition 2.2.9.

Let $g \in O$ and let $\mathfrak{h}$ be a *real* polarization at g. This means that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and that $\mathfrak{h}$ is maximal isotropic with respect to B_g. Let H be the analytic subgroup of G corresponding to $\mathfrak{h}$.

For $l = sg \in O$ we set $\mathfrak{h}_l = \mathrm{Ad}(s)\mathfrak{h}$ and $H_l = sHs^{-1}$. Further, we let $\alpha_l \colon \mathfrak{g}/\mathfrak{g}_l \to T_l(O)$ denote the canonical vector space isomorphism and we set F_l to be the image of $\mathfrak{h}_l/g_l$ by α_l. We have $F_g = \alpha_g(\mathfrak{h}/\mathfrak{g}_g)$, and denoting by $\gamma(s)\colon O \to O$ the map $\gamma(s)\colon l \to sl$ we have $\gamma(s)_* F_l = F_{sl}$. The map $F\colon l \to F_l$ is a G-invariant *real* polarization of the symplectic manifold (O, ω) (cf. [9]).

The group H defines a fibration of the orbit $O \simeq G/G_g$ with base G/H and with the projection given by $\pi\colon l = sg \to sH$. The fiber through $l = sg$

is $\mathscr{L}(l) = \{shg \mid h \in H\} = l + \mathfrak{h}_l^{\perp}$, since $\mathfrak{h}$ satisfies the Pukanszky condition [10]. The vertical subspace through $l \in O$ is F_l.

Let $\mathscr{V}(O)$ denote the set of all smooth *complex* vector fields on O. In [9] we have defined certain Lie subalgebras $\mathscr{V}_F^0(O)$ and $\mathscr{V}_F^1(O)$ of $\mathscr{V}(O)$, and we have defined certain Lie subalgebras $\mathscr{E}_F^0(O)$ and $\mathscr{E}_F^1(O)$ of $\mathscr{E}(O)$ (equipped with Poisson bracket). We set $\mathscr{P}_F^0(O) = \mathscr{E}_F^0(O) \cap \mathscr{P}(O)$ and $\mathscr{P}_F^1(O) = \mathscr{E}_F^1(O) \cap \mathscr{P}(O)$.

It follows from [9] that the map from $\mathscr{E}(G/H)$ to $\mathscr{E}_F^0(O)$ given by $\varphi \to \varphi \circ \pi$ is a linear bijection. Considering polynomial functions we have

Lemma 2.2.1. *The map $\varphi \to \varphi \circ \pi$ maps $\mathscr{P}(G/H)$ onto $\mathscr{P}_F^0(O)$.*

Proof. This follows from Lemma 1.3.2.

We now set
$$e = \{1 \leq j \leq m | X_j \notin \mathfrak{g}_{j-1} + \mathfrak{h}\},$$
and write $e = \{k_1 < \cdots < k_{d/2}\}$.

Lemma 2.2.2. *For any $l \in O$ we have that*
$$e = \{1 \leq j \leq m | X_j \notin \mathfrak{g}_{j-1} + \mathfrak{h}_\ell\}.$$
Proof. Clearly $e = \{1 \leq j \leq m | \mathrm{Ad}(s) X_j \notin \mathfrak{g}_{j-1} + \mathfrak{h}_l\}$, where $l = sg$. But since $\mathrm{Ad}(s) X_j = X_j \pmod{\mathfrak{g}_{j-1}}$ the lemma is proved.

Using the notation from [9, §1.2], we get from Lemma 2.2.2 that $O_e = O$.

Lemma 2.2.3. *Let $l \in O$. Any element $X \in \mathfrak{g}$ can be written uniquely*
$$X = \sum_{u=1}^{d/2} a_{X,u}(l) X_{k_u} + X(l),$$
where $X(l) \in \mathfrak{h}_l$ and $a_{X,u}(l) \in \mathbf{R}$, $u = 1, \ldots, d/2$. The functions $l \to a_{X,u}(l) : O \to \mathbf{R}$ and $l \to X(l) \colon O \to \mathfrak{g}$ are polynomial functions. Moreover, $a_{X,u} \in \mathscr{P}_F^0(O)$ for all $X \in \mathfrak{g}$, $u = 1, \ldots, d/2$, and $X(hl) = X(l)$ for all $l \in O$, $h \in H_l$.

Proof. This is the analogue of Lemma 1.2.3 in [9]. All we have to prove is that $a_{X,u}$ are polynomial functions. To this end we pick $g \in O$, and select a basis $Y_1, \ldots, Y_{d/2}$ of $\mathfrak{h} \pmod{\mathfrak{g}_g}$. We then have with $l = sg$: $B_\ell(\mathrm{Ad}(s) Y_u, X) = \sum_{v=1}^{d/2} B_l(\mathrm{Ad}(s) Y_u, X_{k_v}) a_{X,v}(l)$. Defining $\mathbf{B}(s)$ to be the $d/2 \times d/2$-matrix
$$\begin{aligned}\mathbf{B}(s) &= \left[B_l(\mathrm{Ad}(s) Y_u, X_{k_v})\right]_{1 \leq u, v \leq d/2} \\ &= \left[B_g(Y_u, \mathrm{Ad}(s^{-1}) X_{k_v})\right]_{1 \leq u, v \leq d/2},\end{aligned}$$
$\mathbf{C}(s)$ to be the $d/2 \times 1$-matrix
$$\begin{aligned}\mathbf{C}(s) &= \left[B_l(\mathrm{Ad}(s) Y_u, X)\right]_{1 \leq u \leq d/2} \\ &= \left[B_g(Y_u, \mathrm{Ad}(s^{-1}) X)\right]_{1 \leq u \leq d/2},\end{aligned}$$

and $\mathbf{A}(l)$ to be the $d/2 \times 1$-matrix

$$\mathbf{A}(l) = \left[a_{X,v}(l)\right]_{1 \le v \le d/2}$$

we have that $\mathbf{B}(s)$ is regular, and $\mathbf{C}(s) = \mathbf{B}(s)\mathbf{A}(l)$, hence $\mathbf{A}(l) = \mathbf{B}(s)^{-1}\mathbf{C}(s)$. Now clearly the matrix functions $s \to \mathbf{A}(s)$ and $s \to \mathbf{B}(s)$ have polynomial entries. It therefore suffices to prove that the function $s \to \det \mathbf{B}(s)$ is constant on G. To this end we note that we can write $\mathrm{Ad}(s^{-1})X_{k_v} = \sum_{w=1}^{d/2} t_{wv} X_{k_w} + W_v$, where $W_v \in \mathfrak{h}$, and where $t_{vv} = 1$ while $t_{wv} = 0$ for $w > v$. This means that $\mathbf{B}(s) = \mathbf{B}(e_G)\mathbf{T}$, where e_G is the neutral element in G, and where $\mathbf{T}$ is the matrix $\left[t_{wv}\right]_{1 \le w,v \le d/2}$, and clearly $\det \mathbf{T} = 1$. But this shows that $\det \mathbf{B}(s) = \det \mathbf{B}(e_G)$, and this ends the proof of the lemma.

Recall that $\pi\colon O \to G/H$ is the map $\pi\colon l = sg \to sH$. The map π is a polynomial map (cf. Lemma 2.2.1).

Lemma 2.2.4. *For any $X \in \mathfrak{g}$ we can write ψ^X uniquely as*

$$\psi^X = \sum_{u=1}^{d/2} a_{X,u} \psi^{X_{k_u}} + a_{X,0},$$

where $a_{X,0}, a_{X,1}, \dots, a_{X,d/2} \in \mathscr{P}_F^0(O)$.

Proof. The uniqueness follows from [9, Lemma 1.2.3]. To prove the existence we use Lemma 2.2.3 to write

$$X = \sum_{u=1}^{d/2} a_{X,u}(l) X_{k_u} + X(l)$$

for each $l \in O$, where $X(l) \in \mathfrak{h}_\ell$. Setting $a_{X,0}(l) = \langle l, X(l)\rangle$ we have

$$\psi^X = \sum_{u=1}^{d/2} a_{X,u} \psi^{X_{k_u}} + a_{X,0}.$$

Since already $a_{X,1}, \dots, a_{X,d/2} \in \mathscr{P}_F^0(O)$, we only need to prove that $a_{X,0} \in \mathscr{P}_F^0(O)$. But clearly $a_{X,0}$ is a polynomial function, since $l \to X(l)$ is a polynomial map. For $h \in H_l$ we have $a_{X,0}(hl) = \langle hl, X(hl)\rangle = \langle hl, X(l)\rangle = \langle l, \mathrm{Ad}(h^{-1})X(l)\rangle = \langle l, X(l)\rangle = a_{X,0}(l)$, since $\mathfrak{h}_l$ is a polarization at l. This ends the proof of the lemma.

Lemma 2.2.5. *For each $X \in \mathfrak{g}$ we can write ψ^X uniquely as*

$$\psi^X = \sum_{u=1}^{d/2} b_{X,u} \circ \pi \psi^{X_{k_u}} + b_{X,0} \circ \pi,$$

where $b_{X,0}, b_{X,1}, \dots, b_{X,d/2} \in \mathscr{P}(G/H)$.

Proof. We have from Lemma 2.2.1 that the map $\varphi \to \varphi\circ\pi : \mathscr{P}(G/H) \to \mathscr{P}_F^0(O)$ is a bijection, and the result therefore follows from the previous lemma.

Lemma 2.2.6. *The map* $\Phi : O \to \mathbf{R}^{d/2} \times G/H$ *given by*

$$\Phi : l \to (\langle l, X_{k_i}\rangle, \dots, \langle l, X_{k_{d/2}}\rangle, \pi(l))$$

is a bi-polynomial diffeomorphism.

Proof. It follows from [9, Lemma 1.2.4] that Φ is a diffeomorphism onto $\mathbf{R}^{d/2}\times G/H$, and clearly Φ is a polynomial map. We just have to prove that the inverse of Φ is polynomial. To do this, let $\tau : G \to \mathbf{R}^{d/2}$ be a polynomial chart on G/H, and define the polynomial chart $\tilde{\tau} : \mathbf{R}^{d/2} \times G/H \to \mathbf{R}^{d/2} \times \mathbf{R}^{d/2} = \mathbf{R}^d$ by $\tilde{\tau} = \mathrm{id}\times\tau$. We have to prove that $\Phi^{-1}\circ\tilde{\tau}^{-1} : \mathbf{R}^d \to O$ is a polynomial map, and for this it suffices to prove that the map $(v, w) \to \langle X, \Phi^{-1}\circ\tilde{\tau}^{-1}(v, w)\rangle :$ $\mathbf{R}^d \to \mathbf{R}$ is polynomial for each $X \in \mathfrak{g}$. For $(v, w) \in \mathbf{R}^{d/2} \times \mathbf{R}^{d/2}$, set $l = \Phi^{-1}\circ\tilde{\tau}^{-1}(v, w)$. Then $(\langle l, X_{k_1}\rangle, \dots, \langle l, X_{k_{d/2}}\rangle, \pi(l)) = \Phi(l) = \tilde{\tau}^{-1}(v, w) =$ $(v, \tau^{-1}(w))$. It follows that $v_u = \langle l, X_{k_u}\rangle$ for $1 \le u \le d/2$, and that $\pi(l) =$ $\tau^{-1}(w)$. With the notation from Lemma 2.2.5 we then get that

$$\langle X, \Phi^{-1}\circ\tilde{\tau}^{-1}(v, w)\rangle = \langle l, X\rangle = \sum_{u=1}^{d/2} b_{X,u}(\tau^{-1}(w))v_u + b_{X,0}(\tau^{-1}(w)).$$

Since the functions $b_{X,u}$, $u = 0, 1, \dots, d/2$, are polynomial we have proved the lemma.

Let τ be a polynomial chart on G/H and write $\tau = (x_1, \dots, x_{d/2})$. Define now the functions $y_1, \dots, y_d$ on O by $y_u(l) = \langle l, X_{k_u}\rangle$ and $y_{d/2+u}(l) =$ $x_u(\pi(l))$, and define the map $\sigma : O \to \mathbf{R}^d$ by $\sigma = (y_1, \dots, y_d)$.

Lemma 2.2.7. σ *defines a polynomial chart on* O.

Proof. Since $\sigma(l) = (\langle l, X_{K_1}\rangle, \dots, \langle l, X_{k_{d/2}}\rangle, x_1(\pi(l)), \dots, x_{d/2}(\pi(l)))$, this is immediate from the previous lemma.

Lemma 2.2.8. *We have*

$$\mathscr{P}_F^0(O) = \{\varphi \in \mathscr{P}(O) \mid \partial\varphi/\partial y_u = 0,\ 1 \le u \le d/2\},$$
$$\mathscr{P}_F^1(O) = \{\varphi \in \mathscr{P}(O) \mid \partial^2\varphi/\partial y_u\partial y_v = 0,\ 1 \le u, v \le d/2\}.$$

Proof. This follows from Lemma 2.2.7 and [9, Lemma 1.2.8].

Proposition 2.2.9. *The space* $\mathscr{P}_F^1(O)$ *is a free* $\mathscr{P}_F^0(O)$*-module with basis 1,* $\psi^{X_{k_1}}$, $\dots, \psi^{X_{k_{d/2}}}$.

Proof. This follows immediately from Lemma 2.2.8, cf. [9, Lemma 1.2.9]. This ends the proof of the lemma.

Lemma 2.2.10. *For each* $l \in O$ *we have*

$$F_l = \{(\xi_\varphi)_l \mid \varphi \in \mathscr{P}_F^0(O)\}.$$

Proof. If $\varphi \in \mathscr{P}_F^0(O)$ we have that $(\xi_\varphi)_l \in F_l$ by the definition of $\mathscr{P}_F^0(O)$. Now it follows from [9, Lemma 1.2.6] that the tangent vectors $(\xi_{y_{d/2+1}})_l, \dots, (\xi_{y_d})_l$ span F_l, and since y_u belongs to $\mathscr{P}_F^0(O)$ for $d/2 < u \le d$ we have proved the lemma.

Proposition 2.2.11. *We have*

$$\mathscr{V}_F^0(O) = \{\xi \in \mathscr{V}(O) \mid \xi(\mathscr{P}_F^0(O)) = 0\},$$
$$\mathscr{V}_F^1(O) = \{\xi \in \mathscr{V}(O) \mid \xi(\mathscr{P}_F^0(O)) \subset \mathscr{E}_F^0(O)\}.$$

Proof. This is proved just like Proposition 1.2.16 in [9], using Lemma 2.2.10.

Proposition 2.2.12. *We have*

$$\mathscr{E}_F^0(O) = \{\varphi \in \mathscr{E}(O) \mid \{\varphi, \mathscr{P}_F^0(O)\} = 0\},$$
$$\mathscr{E}_F^1(O) = \{\varphi \in \mathscr{E}(O) \mid \{\varphi, \mathscr{P}_F^0(O)\} \subset \mathscr{E}_F^0(O)\}.$$

Proof. This follows immediately from Proposition 2.2.11.

Proposition 2.2.13. *The space* $\mathscr{P}_F^0(O)$ *is a maximal commutative subalgebra of* $\mathscr{P}(O)$, *and* $\mathscr{P}_F^1(O)$ *is the normalizer of* $\mathscr{P}_F^0(O)$ *in* $\mathscr{P}(O)$.

Proof. This follows immediately from Proposition 2.2.12.

2.3. Polynomial quantized operators. Let now $\chi : H \to \mathbf{T}$ be the continuous homomorphism with $\chi(\exp X) = e^{i\langle g, X\rangle}$ for $X \in \mathfrak{h}$. We know that the space $\mathscr{B}^1(G, \chi)$ in a natural way is an $\mathscr{E}_F^0(O)$-module. It follows from Lemma 2.2.1 that similarly we can consider $\mathscr{A}^1(G, \chi)$ as a $\mathscr{P}_F^0(O)$-module. With this terminology we get from Proposition 1.7.4:

Proposition 2.3.1. *The space* $\mathscr{A}^1(G, \chi)$ *is a free* $\mathscr{P}_F^0(O)$*-module with basis* 1, $d\gamma_\chi(X_{k_1}), \dots, d\gamma_\chi(X_{k_{d/2}})$.

2.4. Quantization and nilpotent groups. We now consider the quantization map $\delta_\chi : \mathscr{E}_F^1(O) \to \mathscr{B}^1(G, \chi)$. As shown in [9], δ_χ is a Lie algebra isomorphism. It follows from Proposition 1.7.4 and Proposition 2.2.9 that the map $\varphi \to \delta_\chi(\varphi)$ carries $\mathscr{P}_F^1(O)$ into $\mathscr{A}^1(G, \chi)$, and, in fact, also *onto* $\mathscr{A}^1(G, \chi)$. We thus have

Theorem 2.4.1. *The map*

$$\delta_\chi : \mathscr{P}_F^1(O) \to \mathscr{A}^1(G, \chi)$$

is a (*Lie-algebra*) *isomorphism.*

2.5. Unitary representations. We now consider the connection with unitary representations. Let μ be an *invariant* measure on G/H and let π_χ be the representation $\mathrm{ind}_{H\uparrow G}\chi$ of G on $L_\mu^2(G, \chi)$. The representation π_χ is in the equivalence class of irreducible representations associated with the orbit O. Associated with μ there is defined an involution $\sharp$ on $\mathscr{E}_F^1(O)$ such that $\delta_\chi(\varphi^\sharp) = \delta_\chi(\varphi)^*$, where $\delta_\chi(\varphi)^*$ is the formal adjoint of $\delta_\chi(\varphi)$ considered as an operator on $\mathscr{D}(G,\chi) \subset L^2_\mu(G,\chi)$ [9, §3].

Lemma 2.5.1. *$\mathscr{P}_F^1(O)$ is invariant under the involution $\sharp$.*

Proof. Since μ is invariant it follows that the function w_χ from [9, §3.1] is equal to 0. Therefore we get from [9, Lemma 3.1.7] that $\lambda(\varphi\psi^X) = \{\psi^X, \varphi\} \in \mathscr{P}_F^0(O)$ for $\varphi \in \mathscr{P}_F^0(O)$, $X \in \mathfrak{g}$. Since $\lambda(\varphi) = 0$ for $\varphi \in \mathscr{P}_F^0(O)$ we get from Proposition 2.2.9 that if $\varphi \in \mathscr{P}_F^1(O)$, then $\lambda(\varphi) \in \mathscr{P}_F^0(O)$. This proves the lemma, since $\varphi^\sharp = -\overline{\varphi} + i\lambda(\overline{\varphi})$ for $\varphi \in \mathscr{E}_F^1(O)$ [9, §3.1].

In [9, §3.1] we have also defined a map $\varepsilon_\chi : \mathscr{E}_F^1(O) \to \mathscr{B}^1(G,\chi)$. Using the previous lemma we get from [9, Theorem 3.1.13]:

Theorem 2.5.2. *The map $\varepsilon_\chi : \mathscr{P}_F^1(O) \to \mathscr{A}^1(G,\chi)$ is a Lie algebra isomorphism, and*

$$\varepsilon_\chi(\varphi)^* = -\varepsilon_\chi(\overline{\varphi})$$

for all $\varphi \in \mathscr{P}_F^1(O)$, and

$$\varepsilon_\chi(\psi^X)f = d\pi(X)f$$

for all $f \in \mathscr{D}(G,\chi)$, $X \in \mathfrak{g}$.

2.6. Global canonical coordinates. We can now prove

Theorem 2.6.1. *There exists functions $q_1, \dots, q_{d/2} \in \mathscr{P}_F^0(O)$, and functions $p_1, \dots, p_{d/2} \in \mathscr{P}_F^1(O)$ such that $(p_1, \dots, p_{d/2}, q_1, \dots, q_{d/2})$ is a set of canonical coordinates on O. Moreover, any such choice of canonical coordinates defines a polynomial chart on O.*

Proof. Since $\mathscr{A}^1(G,\chi)$ and $\mathscr{A}^1(\mathbf{R}^{d/2})$ are isomorphic as Lie algebras, there exist skewadjoint elements $D_1, \dots D_{d/2}$ in $\mathscr{A}^1(G,\chi)$ and selfadjoint elements $M_1, \dots, M_{d/2}$ in $\mathscr{A}^0(G,\chi)$ such that $[D_u, D_v] = 0$, $[M_u, M_v] = 0$ and $[D_u, M_v] = i\delta_{uv}$, $1 \le u, v \le d/2$. From Theorem 2.5.2 we get that there exist *real* functions $p_1, \dots, p_{d/2}$ in $\mathscr{P}_F^1(O)$ such that $\varepsilon_\chi(p_u) = D_u$ and *real* functions $q_1, \dots, q_{d/2}$ in $\mathscr{P}_F^0(O)$ such that $\varepsilon_\chi(q_u) = iM_u$, $1 \le u \le d/2$. We can then show that the set $(p_1, \dots, p_{d/2}, q_1, \dots, q_{d/2})$ defines canonical coordinates just like in [9, proof of Theorem 2.2.1].

Let us then show that any such choice $\sigma = (p_1, \dots, p_{d/2}, q_1, \dots, q_{d/2})$ of canonical coordinates defines a polynomial chart. Set $U = \sigma(O)$ (U will be shown to be equal to $\mathbf{R}^d$ in the following). We have to prove that the map $\sigma^{-1}: U \to O$ is a polynomial map. To do this, it suffices to show that the map

$(v, w) \to \langle X, \sigma^{-1}(v, w)\rangle : U \to \mathbf{R}$ is a polynomial map. We write

$$\psi^X = \sum_{u=1}^{d/2} a_{X,u}(q)p_u + a_{X,0}(q).$$

Here $a_{X,0}, a_{X,1}, \ldots, a_{X,d/2}$ are polynomial functions, since $a_{X,u} = \{\psi^X, q_u\}$ (Lemma 2.1.1). Now if $l = \sigma^{-1}(v, w)$, then $p_u(l) = v_u$ and $q_u(l) = w_u$, and it follows that

$$\langle X, \sigma^{-1}(v, w)\rangle = \langle X, l\rangle = \sum_{u=1}^{d/2} a_{X,u}(v)w_u + a_{X,0}(v).$$

This shows that the map $(v, w) \to \langle X, \sigma^{-1}(v, w)\rangle$ is a polynomial function, and this ends the proof of the theorem.

Remark 2.6.2. *Although it is known that there exists global canonical coordinates on* O ([1], *cf.* [9]) *the result in the previous theorem is considerably stronger. What we have shown here is that for any choice of a real polarization there exists polynomial global canonical coordinates compatible with this polarization.*

2.7. Construction of unitary representations using global canonical coordinates. Let $p_1, \ldots, p_{d/2}, q_1, \ldots, q_{d/2}$ be global canonical coordinates on O such that $q_1, \ldots, q_{d/2} \in \mathscr{P}_F^0(O)$, and $p_1, \ldots, p_{d/2} \in \mathscr{P}_F^1(O)$. Such coordinates exists by Theorem 2.6.1, and, moreover, they define a polynomial chart on O.

We define the function $\tau : G/H \to \mathbf{R}^{d/2}$ by $\tau(sH) = (q_1(sg), \ldots, q_{d/2}(sg))$. Clearly τ is well defined since $q_1, \ldots, q_{d/2}$ belong to $\mathscr{P}_F^0(O)$, and τ is a global chart [9, Lemma 3.2.1] whose image is $\mathbf{R}^{d/2}$. It is easily seen that the quasi-invariant measure μ defined by τ [9, §3.2] is in fact *invariant.*

Lemma 2.7.1. *The polynomial quantizable functions* $\mathscr{P}_F^1(O)$ *are the functions* φ *of the form*

$$\varphi = \sum_{u=1}^{d/2} a_u(q)p_u + a_0(q),$$

where $a_0, a_1, \ldots, a_{d/2} \in \mathscr{P}(\mathbf{R}^{d/2})$.

Proof. This is clear, since $(p_1, \ldots, p_{d/2}, q_1, \ldots, q_{d/2})$ is a polynomial chart.

It follows from Lemma 2.7.1 that we in particular can write

$$\psi^X = \sum_{u=1}^{d/2} a_{X,u}(q)p_u + a_{X,0}(q)$$

for each $X \in \mathfrak{g}$, where $a_{X,u} \in \mathscr{P}(\mathbf{R}^{d/2})$, $u = 0, 1, \ldots, d/2$.

From [9, Theorem 3.2.3] we can then derive

Theorem 2.7.2. *There exists one and only one strongly continuous, unitary representation π of G on $L^2(\mathbf{R}^{d/2})$ such that $\mathscr{D}(\mathbf{R}^{d/2}) \subset H_\pi^\infty$, and such that*

$$d\pi(X)f(t) = \sum_{u=1}^{d/2} a_{X,u}(t)\frac{\partial f}{\partial t_u}(t) + ia_{X,0}(t)f(t) + \frac{1}{2}\sum_{u=1}^{d/2}\frac{\partial a_{X,u}}{\partial t_u}(t)f(t)$$

for $f \in \mathscr{D}(\mathbf{R}^{d/2})$, $X \in \mathfrak{g}$. The representation π is equivalent to π_χ, i.e., it is in the equivalence class of irreducible representations associated with the orbit O.

3. Quantization and the universal enveloping algebra

In the following we shall consider the set of elements in the universal enveloping algebra which are mapped into the set of quantized operators by a given representation.

3.1. The map $\tilde{\Phi}$. Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $g \in \mathfrak{g}^*$, and let $\mathfrak{h}$ be a real polarization at g. Let $\chi: H = \exp\mathfrak{h} \to \mathbf{T}$ be the unitary character with $\chi(\exp X) = e^{i\langle g, X\rangle}$, $X \in \mathfrak{h}$, and set $\pi_\chi = \mathrm{ind}_{H\uparrow G}\chi$. Then π_χ is the irreducible representation associated with the orbit $O = Gg$. We have with the notation from §1.7:

Theorem 3.1.1 (Kirillov). *The differentiable vectors for π_χ are precisely $\mathscr{S}(G, \chi)$, and the image of $U(\mathfrak{g}_{\mathbf{C}})$ by $d\pi_\chi$ is $\mathscr{A}(G, \chi)$.*

This result is due to Kirillov [5]. More specifically he showed that there exists for each $g \in \mathfrak{g}^*$ a polarization $\mathfrak{h}$ at g such that the result holds. It was observed in [2] that it is actually true for any real polarization $\mathfrak{h}$ at g.

We set

$$I(O) = \ker(d\pi_\chi)$$

which is a selfadjoint primitive ideal in $U(\mathfrak{g}_{\mathbf{C}})$ ([4]). The ideal $I(O)$ is equal to the annahilator of the left $U(\mathfrak{g}_{\mathbf{C}})$-module $U(\mathfrak{g}_{\mathbf{C}}) \otimes_{U(\mathfrak{h}_{\mathbf{C}})} \mathbf{C}$ [4]. In [7] I have given an explicit formula for $I(O)$. In the following we shall develop further the ideas put forward in [7]. Let F be the G-invariant polarization of O defined by $\mathfrak{h}$, cf. [9], and set

$$J_F^0(O) = \{u \in U(\mathfrak{g}_{\mathbf{C}}) \mid d\pi_\chi(u) \in \mathscr{A}^0(G, \chi)\}$$

and

$$J_F^1(O) = \{u \in U(\mathfrak{g}_{\mathbf{C}}) \mid d\pi_\chi(u) \in \mathscr{A}^1(G, \chi)\}.$$

(It is easily seen that the definition of $J_F^0(O)$ and $J_F^1(O)$ depends only on F.) In other words, $J_F^0(O)$ ($J_F^1(O)$) is the set of elements in $U(\mathfrak{g}_{\mathbf{C}})$ which are mapped by $d\pi_\chi$ to multiplication operators (differential operators of order at most one). Obviously we have

$$I(O) \subset J_F^0(O) \subset J_F^1(O),$$

and by Kirillov's theorem we have

$$d\pi_\chi(J_F^0(O)) = \mathscr{A}^0(G,\chi)$$

and

$$d\pi_\chi(J_F^1(O)) = \mathscr{A}^1(G,\chi).$$

Factoring $d\pi_\chi$ through $I(O)$ we get a Lie algebra isomorphism $\tilde{d\pi}_\chi$:

$$J_F^1(O)/I(O) \xrightarrow{\tilde{d\pi}_\chi} \mathscr{A}^1(G,\chi)$$

such that

$$J_F^0(O)/I(O) \xrightarrow{\tilde{d\pi}_\chi} \mathscr{A}^0(G,\chi).$$

Now the quantization map

$$\delta_\chi : \mathscr{P}_F^1(O) \to \mathscr{A}^1(G,\chi)$$

is also a Lie algebra isomorphism (Theorem 2.4.1). This means that there is a Lie algebra isomorphism $\tilde{\Phi} : \mathscr{P}_F^1(O) \to J_F^1(O)/I(O)$ which makes the following diagram commutative:

$$\begin{tikzcd} \mathscr{P}_F^1(O) \arrow[rr, "\tilde{\Phi}"] \arrow[dr, "\delta_\chi"'] & & J_F^1(O)/I(O) \arrow[dl, "\tilde{d\pi}_\chi"] \\ & \mathscr{A}^1(G,\chi) & \end{tikzcd}$$

The goal of the following is to give *an explicit description of* $\tilde{\Phi}$ *in the case where* $\mathfrak{h}$ *is a Vergne polarization.* Specifically, we shall construct for a given Jordan-Hölder basis $\mathscr{B}$: $X_1, \ldots, X_m$ and associated Vergne polarization $\mathfrak{h}$ an explicit map $\Phi = \Phi_O^{\mathscr{B}} : \mathscr{P}_F^1(O) \to J_F^1(O)$, F being the polarization of O associated with $\mathfrak{h}$, such that Φ composed with the quotient map $J_F^1(O) \to J_F^1(O)/I(O)$ is the map $\tilde{\Phi}$.

Let $\mathscr{B} : X_1, \ldots, X_m$ be a Jordan-Hölder basis in $\mathfrak{g}$, and set

$$\mathfrak{g}_j = \operatorname{span}(X_1, \ldots, X_j), \qquad 1 \le j \le m.$$

Then

$$\mathscr{F} : \mathfrak{g} = \mathfrak{g}_m \supset \mathfrak{g}_{m-1} \supset \cdots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0 = \{0\}$$

is a Jordan-Hölder sequence in $\mathfrak{g}$. Let $g \in \mathfrak{g}^*$ and set $g_j = g|\mathfrak{g}_j$ for each $1 \le j \le m$. The linear subspace

$$\mathfrak{h}(g,\mathscr{F}) = \sum_{j=1}^m (\mathfrak{g}_j)_{g_j}$$

is then a polarization at $g \in \mathfrak{g}^*$, the so-called *Vergne polarization* at $g \in \mathfrak{g}^*$ associated with $\mathscr{F}$ [3]. We shall call the associated polarization of O for $F(O,\mathscr{F})$, cf. [9]. We set $H(g,\mathscr{F}) = \exp \mathfrak{h}(g,\mathscr{F})$.

Let $\sigma = \sigma_O^{\mathscr{B}} : O \to \mathbf{R}^d$ be the $\mathscr{B}$-chart of O, see §1.6. For each $\varphi \in \mathscr{P}(O)$ the function $\varphi \circ \sigma^{-1}$ is a polynomial in $\mathbf{C}[y_1, \ldots, y_d]$. The element

$\varphi \circ \sigma^{-1}(-iX_{j_1}, \dots, -iX_{j_d})$ is then well defined in $S(\mathfrak{g}_{\mathbf{C}})$, the symmetric algebra of $\mathfrak{g}_{\mathbf{C}}$. For $\varphi \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$ we set

$$\Phi(\varphi) = \Phi_O^{\mathscr{B}}(\varphi) = i\omega(\varphi \circ \sigma^{-1}(-iX_{j_1}, \dots, -iX_{j_d})),$$

where $\omega : S(\mathfrak{g}_{\mathbf{C}}) \to U(\mathfrak{g}_{\mathbf{C}})$ is the symmetrization map. If $d = \dim O = 0$ we set $\Phi(\varphi) = i\varphi(O)\mathbf{1}$, where $\mathbf{1}$ is the unit in $U(\mathfrak{g}_{\mathbf{C}})$.

Theorem 3.1.2. *For $\varphi \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$ we have $d\pi_\chi(\Phi(\varphi)) = \delta_\chi(\varphi)$.*

It follows from the theorem that the value of $\tilde{\Phi}$ on $\varphi \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$ is the canonical image of $\Phi(\varphi)$ in $J^0_F(O)/I(O)$.

Proof of theorem. The proof is by induction on the dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 1$ the result is true by definition. Suppose then that the result has been proved for all dimensions less than m, and that $\dim \mathfrak{g} = m$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and set $\mathfrak{z}_0 = \ker g|\mathfrak{z}$. Suppose first that we are in

Case (a): $\dim \mathfrak{z}_0 > 0$. Set $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}_0$, and let $c : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be the quotient map. We let also c denote the quotient map $c : G \to \tilde{G} = G/Z_0$, where $Z_0 = \exp \mathfrak{z}_0$. Let $\tilde{g}$ be the element in $\tilde{\mathfrak{g}}^*$ such that $\tilde{g} \circ c = g$.

We now define a Jordan-Hölder sequence in $\tilde{\mathfrak{g}}$. We set $I = \{1 \le j \le m \mid \mathfrak{g}_j \not\subset \mathfrak{g}_{j-1} + \mathfrak{z}_0\}$, and write $I = \{i_1 < \cdots < i_n\}$ and $\tilde{\mathfrak{g}}_r = (\mathfrak{g}_{i_r} + \mathfrak{z}_0)/\mathfrak{z}_0$, $r = 1, \dots, n$. Then

$$\tilde{\mathscr{F}} : \tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_n \supset \tilde{\mathfrak{g}}_{n-1} \supset \cdots \supset \tilde{\mathfrak{g}}_1 \supset \tilde{\mathfrak{g}}_0 = \{0\}$$

is a Jordan-Hölder sequence in $\tilde{\mathfrak{g}}$, and setting $\tilde{X}_r = c(X_{i_r})$ we have that $\tilde{\mathscr{B}} : \tilde{X}_1, \dots, \tilde{X}_n$ is a Jordan-Hölder basis in $\tilde{\mathfrak{g}}$ compatible with $\tilde{\mathscr{F}}$. We note that $\mathfrak{z}_0 \subset \mathfrak{g}_g$, and that $\tilde{\mathfrak{g}}_{\tilde{g}} = \mathfrak{g}_g/\mathfrak{z}_0$.

For $1 \le j \le m$ we define the subalgebra $\mathfrak{g}_j^\sharp$ of $\mathfrak{g}$ by $\mathfrak{g}_j^\sharp = \mathfrak{g}_j + \mathfrak{z}_0$, and we set $g_j^\sharp = g|\mathfrak{g}_j^\sharp$. It is then immediate that $(\mathfrak{g}_j^\sharp)_{g_j^\sharp} = (\mathfrak{g}_j)_{g_j} + \mathfrak{z}_0$. Suppose now that $i_r \le j < i_{r+1}$, $1 \le r < n$. Then clearly $\mathfrak{g}_{i_r}^\sharp \subset \mathfrak{g}_j^\sharp$, but we also have that $\mathfrak{g}_j \subset \mathfrak{g}_{i_r} + \mathfrak{z}_0$, and therefore that $\mathfrak{g}_j + \mathfrak{z}_0 \subset \mathfrak{g}_{i_r} + \mathfrak{z}_0$, so $\mathfrak{g}_{i_r}^\sharp = \mathfrak{g}_j^\sharp$. We next note that since $\mathfrak{z}_0 \subset \mathfrak{g}_g \subset \mathfrak{h}(g, \mathscr{F})$ we have that $\mathfrak{h}(g, \mathscr{F}) = \mathfrak{h}(g, \mathscr{F}) + \mathfrak{z}_0 = \sum_{j=1}^m (\mathfrak{g}_j)_{g_j} + \mathfrak{z}_0 = \sum_{j=1}^m ((\mathfrak{g}_j)_{g_j} + \mathfrak{z}_0) = \sum_{j=1}^m (\mathfrak{g}_j^\sharp)_{g_j^\sharp} = \sum_{r=1}^n (\mathfrak{g}_{i_r}^\sharp)_{g_{i_r}^\sharp}$. For each $1 \le r \le n$ we have that $c(\mathfrak{g}_{i_r}^\sharp) = \tilde{\mathfrak{g}}_r$, and $\tilde{g}_r \circ (c|\mathfrak{g}_{i_r}^\sharp) = g_{i_r}^\sharp$. Since $\mathfrak{z}_0 \subset (\mathfrak{g}_{i_r}^\sharp)_{g_{i_r}^\sharp}$ we find that $(\mathfrak{g}_{i_r}^\sharp)_{g_{i_r}^\sharp}/\mathfrak{z}_0 = (\tilde{\mathfrak{g}}_r)_{\tilde{g}_r}$. The conclusion is that $c(\mathfrak{h}(g, \mathscr{F})) = \sum_{r=1}^n c((\mathfrak{g}_{i_r}^\sharp)_{g_{i_r}^\sharp}) = \sum_{r=1}^n (\tilde{\mathfrak{g}}_r)_{\tilde{g}_r} = \mathfrak{h}(\tilde{g}, \tilde{\mathscr{F}})$, and therefore that $\mathfrak{h}(\tilde{g}, \tilde{\mathscr{F}}) = \mathfrak{h}(g, \mathscr{F})/\mathfrak{z}_0$, and that $\tilde{H} = H/Z_0$, where $\tilde{H} = H(\tilde{g}, \tilde{\mathscr{F}})$ and $H = H(g, \mathscr{F})$.

We now consider the orbits $O = Gg$ and $\tilde{O} = \tilde{G}\tilde{g}$. The map $\tilde{O} \to O$ given by $\tilde{l} \to \tilde{l} \circ c = l$ is a bijection. Also, if $\varphi \in \mathscr{P}(O)$, then the function $\tilde{\varphi}$ on $\tilde{O}$ defined by $\tilde{\varphi}(\tilde{l}) = \varphi(l \circ c)$ is in $\mathscr{P}(\tilde{O})$; in fact, for $X \in \mathfrak{g}$ we have $(\psi^X)^\sim = \psi^{\tilde{X}}$, where $\tilde{X} = c(X)$, and the assertion then follows from Lemma 1.7.2. It follows that the map $\varphi \to \tilde{\varphi}$ is a linear bijection from $\mathscr{P}(O)$ onto $\mathscr{P}(\tilde{O})$. For $s \in G$, $h \in H$, let $\tilde{s} = c(s)$, $\tilde{h} = c(h)$. We have $\tilde{\varphi}(\tilde{s}\tilde{h}\tilde{g}) = \varphi(shg)$, so $\varphi \to \tilde{\varphi}$ maps $\mathscr{P}^0_{F(O,\mathscr{F})}(O)$ bijectively onto $\mathscr{P}^0_{F(\tilde{O},\tilde{\mathscr{F}})}(\tilde{O})$.

Let $\tilde{\chi}\colon \tilde{G} \to \mathbf{T}$ be the unitary character with $\tilde{\chi}(\exp \tilde{X}) = e^{i\langle \tilde{g}, \tilde{X}\rangle}$, $\tilde{X} \in \mathfrak{h}(\tilde{g}, \tilde{\mathscr{F}})$. Then $\tilde{\chi} \circ c = \chi$.

There is defined a map $V\colon L^2(G,\chi) \to L^2(\tilde{G},\tilde{\chi})$ by $f \to \tilde{f}$, where $\tilde{f} = f \circ c$. The map V is clearly unitary and $V\pi_\chi(s)V^{-1} = \pi_{\tilde{\chi}}(\tilde{s})$, where $\pi_\chi = \operatorname{ind}_{H\uparrow G}\chi$, $\pi_{\tilde{\chi}} = \operatorname{ind}_{\tilde{H}\uparrow\tilde{G}}\tilde{\chi}$, and $\tilde{s} = c(s)$ for $s \in G$. It is immediately seen that V maps $\mathscr{S}(G,\chi)$ onto $\mathscr{S}(\tilde{G},\tilde{\chi})$. Moreover, $\delta_{\tilde{\chi}}(\tilde{\varphi})\tilde{f}(\tilde{s}) = \tilde{\varphi}(\tilde{s}\tilde{g})\tilde{f}(\tilde{s}) = \varphi(sg)f(s) = \delta_\chi(\varphi)f(s)$, so $V\delta_\chi(\varphi)f = \delta_{\tilde{\chi}}(\tilde{\varphi})Vf$, or $V\delta_\chi(\varphi)V^{-1} = \delta_{\tilde{\chi}}(\tilde{\varphi})$.

Now we have to prove that $d\pi_\chi(\Phi^{\mathscr{B}}_O(\varphi)) = \delta_\chi(\varphi)$, or, equivalently,

$$d(\pi_{\tilde{\chi}} \circ c)(\Phi^{\mathscr{B}}_O(\varphi)) = \delta_{\tilde{\chi}}(\tilde{\varphi}) \quad \text{or} \quad d\pi_{\tilde{\chi}}(c(\Phi^{\mathscr{B}}_O(\varphi)) = \delta_{\tilde{\chi}}(\tilde{\varphi}).$$

By the induction hypothesis we have $d\pi_{\tilde{\chi}}(\Phi^{\tilde{\mathscr{B}}}_{\tilde{O}}(\tilde{\varphi})) = \delta_{\tilde{\chi}}(\tilde{\varphi})$, so we will be done if we can show that $c(\Phi^{\mathscr{B}}_O(\varphi)) = \Phi^{\tilde{\mathscr{B}}}_{\tilde{O}}(\tilde{\varphi})$.

To show that $c(\Phi^{\mathscr{B}}_O(\varphi)) = \Phi^{\tilde{\mathscr{B}}}_{\tilde{O}}(\tilde{\varphi})$ we start by noting that $J_g \subset I$, since $j \notin I \Rightarrow X_j \in \mathfrak{g}_{j-1} + \mathfrak{z}_0 \Rightarrow X_j \in \mathfrak{g}_{j-1} + \mathfrak{g}_g \Rightarrow j \notin J_g$. Writing $e = J_g = \{j_1 < \cdots < j_d\}$ and $\tilde{e} = J_{\tilde{g}} = \{\tilde{j}_1 < \cdots < \tilde{j}_d\}$ we have that $i_{\tilde{j}_u} = j_u$ for $u = 1, \ldots, d$.

Let $\sigma^{\mathscr{B}}_O$ be the $\mathscr{B}$-chart of O, and let $\sigma^{\tilde{\mathscr{B}}}_{\tilde{O}}$ be the $\tilde{\mathscr{B}}$-chart of $\tilde{O}$. We have $\sigma^{\mathscr{B}}_O(\tilde{l} \circ c)_u = \langle \tilde{l} \circ c, X_{j_u}\rangle = \langle \tilde{l} \circ c, X_{i_{\tilde{j}_u}}\rangle = \langle \tilde{l}, c(X_{i_{\tilde{j}_u}})\rangle = \langle \tilde{l}, \tilde{X}_{\tilde{j}_u}\rangle = \sigma^{\tilde{\mathscr{B}}}_{\tilde{O}}(\tilde{l})_u$, so $\sigma^{\mathscr{B}}_O(\tilde{l} \circ c) = \sigma^{\tilde{\mathscr{B}}}_{\tilde{O}}(\tilde{l})$. Therefore, if $y \in \mathbf{R}^d$, and $\tilde{l} = \sigma^{\tilde{\mathscr{B}}^{-1}}_{\tilde{O}}(y)$, then $\sigma^{\mathscr{B}}_O(\tilde{l} \circ c) = y$. It follows that $\tilde{\varphi} \circ \sigma^{\tilde{\mathscr{B}}^{-1}}_{\tilde{O}}(y) = \tilde{\varphi}(\tilde{l}) = \varphi(\tilde{l} \circ c) = \varphi \circ \sigma^{\mathscr{B}^{-1}}_O(y)$, so $\tilde{\varphi} \circ \sigma^{\tilde{\mathscr{B}}^{-1}}_{\tilde{O}} = \varphi \circ \sigma^{\mathscr{B}^{-1}}_O$. Therefore

$$\varphi \circ \sigma^{\mathscr{B}^{-1}}_O(-iX_{j_1}, \ldots, -iX_{j_d}) = \tilde{\varphi} \circ \sigma^{\tilde{\mathscr{B}}^{-1}}_{\tilde{O}}(-iX_{i_{\tilde{j}_1}}, \ldots, -iX_{i_{\tilde{j}_d}})$$

which implies $c(\varphi \circ \sigma^{\mathscr{B}^{-1}}_O(-iX_{j_1}, \ldots, -iX_{j_d})) = \tilde{\varphi} \circ \sigma^{\tilde{\mathscr{B}}^{-1}}_{\tilde{O}}(-i\tilde{X}_{\tilde{j}_1}, \ldots, -i\tilde{X}_{\tilde{j}_d})$, and taking symmetrization ω and using that c commutes with ω we get $c(\Phi^{\mathscr{B}}_O(\varphi)) = \Phi^{\tilde{\mathscr{B}}}_{\tilde{O}}(\tilde{\varphi})$. This finishes case (a).

Suppose then that we are in

Case (b): $\dim \mathfrak{z}_0 = 0$. Then $\dim \mathfrak{z} = 1$ and $\langle g, \mathfrak{z}\rangle \neq 0$, so $\mathfrak{z} = \mathfrak{g}_1$, and $\langle g, X_1\rangle \neq 0$. In particular $[\mathfrak{g}, \mathfrak{g}_2] = \mathfrak{g}_1$, and therefore $\mathfrak{g}_2 \not\subset \mathfrak{g}_g$, hence $2 \in J_g = e$, and $j_1 = 2$. Note also that $\mathfrak{g}_g \subset \mathfrak{k} = \ker \operatorname{ad} X_2$ (since otherwise $\mathfrak{g}_g + \mathfrak{k} = \mathfrak{g}$ and therefore $\langle g, \mathfrak{g}_1\rangle = \langle g, [\mathfrak{g}, X_2]\rangle = \langle \mathfrak{g}g, X_2\rangle = \langle \mathfrak{k}g, X_2\rangle = 0$ which is a contradiction). We then claim that we can assume that $\mathfrak{g}_{m-1} = \mathfrak{k} = \ker \operatorname{ad} X_2$.

Proof of claim. Clearly $\mathfrak{k} = \ker \operatorname{ad} X_2$ is an ideal in $\mathfrak{g}$ of codimension 1. Set $p = \min\{1 \leq j \leq m \mid X_j \notin \mathfrak{k}\}$. Then p is well defined, $p \geq 3$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathbf{R}X_p$.

It is easily seen that $p \in J_g$ (in fact, if $p \notin J_g$, then $X_p \in \mathfrak{g}_g + \mathfrak{g}_{p-1} \subset \mathfrak{g}_g + \mathfrak{k} \subset \mathfrak{k}$ which is a contradiction). We then define a new basis $\hat{\mathscr{B}}$: $\hat{X}_1, \ldots, \hat{X}_m$ in $\mathfrak{g}$ in the following way: For $1 \leq j \leq p-1$ we set $\hat{X}_j = X_j$, for $p \leq j \leq m-1$ we set $\hat{X}_j = X_{j+1} + c_{j+1}X_p$, where the $c_{j+1} \in \mathbf{R}$ are selected such that $\hat{X}_j \in \mathfrak{k}$ (this is possible since $\mathbf{R}X_p \oplus \mathfrak{k} = \mathfrak{g}$), and finally we set $\hat{X}_m = X_p$. We then define the linear subspaces $\hat{\mathfrak{g}}_j$, $j = 1, \ldots, m$, in $\mathfrak{g}$ by

$$\hat{\mathfrak{g}}_j = \mathbf{R}\hat{X}_1 \oplus \cdots \oplus \mathbf{R}\hat{X}_j.$$

We have $\hat{\mathfrak{g}}_j = \mathfrak{g}_j$ for $1 \leq j \leq p-1$ and $\mathfrak{g}_{j+1} = \hat{\mathfrak{g}}_j \oplus \mathbf{R}X_p$ for $p-1 \leq j \leq m-1$ implying that $\hat{\mathfrak{g}}_j = \mathfrak{g}_{j+1} \cap \mathfrak{k}$ for $p-1 \leq j \leq m-1$. This shows that $\hat{\mathscr{F}}: \hat{\mathfrak{g}}_1, \ldots, \hat{\mathfrak{g}}_m$ is a Jordan-Hölder sequence in $\mathfrak{g}$, and $\hat{\mathscr{B}}$ is compatible with $\hat{\mathscr{F}}$ by construction. By construction we also have $\hat{\mathfrak{g}}_{m-1} = \mathfrak{k}$. We designate the objects associated with this new Jordan-Hölder sequence $\hat{J}_g$, $\hat{e}$, $\mathfrak{h}(O, \hat{\mathscr{F}})$ etc. We write $\hat{J}_g = \hat{e} = \{\hat{\jmath}_1 < \cdots < \hat{\jmath}_d\}$.

For $1 \leq j \leq p-1$ we clearly have that $j \in J_g \Leftrightarrow j \in \hat{J}_g$. Furthermore $p \in J_g$ and $m \in \hat{J}_g$. It is easily seen that if $j_\alpha = p$ we have

$$\begin{aligned} \hat{\jmath}_r &= j_r, \quad 1 \leq r \leq \alpha - 1, \\ \hat{\jmath}_r + 1 &= j_{r+1}, \quad \alpha \leq r \leq d-1, \\ \hat{\jmath}_d &= m. \end{aligned}$$

We next prove that

$$\mathfrak{h}(g, \mathscr{F}) = \mathfrak{h}(g, \hat{\mathscr{F}}).$$

We start by observing that $(\mathfrak{g}_j)_{g_j} \subseteq \mathfrak{k}$ for $1 \leq j \leq m$. In fact, let $\gamma : \mathfrak{g} \to \mathbf{R}$ be the linear map such that $[X, X_2] = \gamma(X)X_1$. Then $X \in \mathfrak{k} \Leftrightarrow \gamma(X) = 0$. If $j = 1$, then clearly $(\mathfrak{g}_j)_{g_j} \subseteq \mathfrak{k}$. Suppose $j \geq 2$. Then $X_2 \in \mathfrak{g}_j$, and if $X \in (\mathfrak{g}_j)_{g_j}$, then $0 = \langle g, [X, X_2] \rangle = \gamma(X)\langle g, X_1 \rangle$ which implies that $\gamma(X) = 0$, hence $X \in \mathfrak{k}$. This shows that $(\mathfrak{g}_j)_{g_j} \subseteq \mathfrak{k}$ for all $1 \leq j \leq m$. Now for $1 \leq j < p$ we have $\hat{\mathfrak{g}}_j = \mathfrak{g}_j$ and therefore $(\hat{\mathfrak{g}}_j)_{\hat{g}_j} = (\mathfrak{g}_j)_{g_j}$. For $p-1 \leq j \leq m-1$ we have $\hat{\mathfrak{g}}_j = \mathfrak{g}_{j+1} \cap \mathfrak{k}$. Therefore, if $X \in (\mathfrak{g}_{j+1})_{g_{j+1}}$ we have that $X \in \mathfrak{k}$, hence $X \in \hat{\mathfrak{g}}_j$, and $0 = \langle g, [X, \mathfrak{g}_{j+1}] \rangle \supset \langle g, [X, \hat{\mathfrak{g}}_j] \rangle$ which shows that $X \in (\hat{\mathfrak{g}}_j)_{\hat{g}_j}$. We conclude that $(\mathfrak{g}_{j+1})_{g_{j+1}} \subset (\hat{\mathfrak{g}}_j)_{\hat{g}_j}$. But then $\mathfrak{h}(g, \mathscr{F}) = \sum_{j=1}^m (\mathfrak{g}_j)_{g_j} = \sum_{j=1}^{p-1} (\mathfrak{g}_j)_{g_j} + \sum_{j=p}^m (\mathfrak{g}_j)_{g_j} = \sum_{j=1}^{p-1} (\hat{g}_j)_{\hat{g}_j} + \sum_{j=p-1}^{m-1} (\mathfrak{g}_{j+1})_{g_{j+1}} \subset \sum_{j=1}^{p-1} (\hat{\mathfrak{g}}_j)_{\hat{g}_j} + \sum_{j=p-1}^{m-1} (\hat{\mathfrak{g}}_j)_{\hat{g}_j} = \sum_{j=1}^{m-1} (\hat{\mathfrak{g}}_j)_{\hat{g}_j} \subset \sum_{j=1}^m (\hat{\mathfrak{g}}_j)_{\hat{g}_j} = \mathfrak{h}(O, \hat{\mathscr{F}})$. This shows that $\mathfrak{h}(g, \mathscr{F}) \subset \mathfrak{h}(O, \hat{\mathscr{F}})$, since $\dim \mathfrak{h}(g, \mathscr{F}) = \dim \mathfrak{h}(O, \hat{\mathscr{F}})$ $(= d/2)$ we have that $\mathfrak{h}(g, \mathscr{F}) = \mathfrak{h}(O, \hat{\mathscr{F}})$.

We shall then compare the $\mathscr{B}$-chart $\sigma_O^{\mathscr{B}}$ and the $\hat{\mathscr{B}}$-chart $\sigma_O^{\hat{\mathscr{B}}}$: For $1 \leq r \leq \alpha - 1$ we have $\sigma_O^{\mathscr{B}}(l)_r = \langle l, X_{j_r} \rangle = \langle l, X_{\hat{\jmath}_r} \rangle = \langle l, \hat{X}_{\hat{\jmath}_r} \rangle = \sigma_O^{\hat{\mathscr{B}}}(l)_r$. For $\alpha \leq r \leq d-1$ we have $\sigma_O^{\hat{\mathscr{B}}}(l)_r = \langle l, \hat{X}_{\hat{\jmath}_r} \rangle = \langle l, X_{\hat{\jmath}_r+1} + c_{\hat{\jmath}_r+1}X_p \rangle = \langle l, X_{j_{r+1}} + c_{j_{r+1}}X_p \rangle = \sigma_O^{\mathscr{B}}(l)_{r+1} + c_{j_{r+1}}\sigma_O^{\mathscr{B}}(l)_\alpha$. Finally $\sigma_O^{\mathscr{B}}(l)_\alpha = \langle l, X_{j_\alpha} \rangle = \langle l, X_p \rangle = \langle l, \hat{X}_m \rangle =$

$\sigma_O^{\hat{\mathscr{B}}}(l)_m$. In other words, if $\sigma_O^{\mathscr{B}}(l) = (y_1, \ldots, y_d)$, and $\sigma_O^{\hat{\mathscr{B}}}(l) = (\hat{y}_1, \ldots, \hat{y}_d)$ we have

$$\begin{aligned}
\hat{y}_1 &= y_1 \\
&\vdots \\
\hat{y}_{\alpha-1} &= y_{\alpha-1} \\
\hat{y}_\alpha &= y_{\alpha+1} + c_{j_{\alpha+1}} y_\alpha \\
&\vdots \\
\hat{y}_r &= y_{r+1} + c_{j_{r+1}} y_\alpha \\
&\vdots \\
\hat{y}_{d-1} &= y_d + c_{j_d} y_\alpha \\
\hat{y}_d &= y_\alpha .
\end{aligned}$$

Let now $\varphi \in \mathscr{P}^0_{F(O,\mathscr{F})}(O) = \mathscr{P}^0_{F(O,\hat{\mathscr{F}})}(O)$. With the above notation we have $\varphi \circ \sigma_O^{\hat{\mathscr{B}}^{-1}}(\hat{y}) = \varphi \circ \sigma_O^{\mathscr{B}^{-1}}(y)$. We now assert that $\Phi_O^{\mathscr{B}}(\varphi) = \Phi_O^{\hat{\mathscr{B}}}(\varphi)$. But this is clear since

$$\varphi \circ \sigma_O^{\hat{\mathscr{B}}^{-1}}(-i\hat{X}_{j_1}, \ldots, -i\hat{X}_{j_d}) = \varphi \circ \sigma_O^{\mathscr{B}^{-1}}(-iX_{j_1}, \ldots, -iX_{j_d}).$$

From now on we then assume that $\mathfrak{g}_{m-1} = \mathfrak{k} = \ker(\mathrm{ad} X_2)$, and set $g_0 = g|\mathfrak{k} = g|\mathfrak{g}_{m-1}$. Finally we set $K = \exp \mathfrak{k}$.

Set $K_g = \{1 \le j \le m \mid X_j \notin \mathfrak{g}_{j-1} + \mathfrak{h}(g,\mathscr{F})\}$. Then writing $K_g = \{k_1 < \cdots < k_{d/2}\}$ we have that $X_{k_1}, \ldots, X_{k_{d/2}}$ is a basis of $\mathfrak{g}$ (mod $\mathfrak{h}(g,\mathscr{F})$) of subalgebra type. Since $\mathfrak{h}(g,\mathscr{F}) \subset \mathfrak{k} = \mathfrak{g}_{m-1}$ we have that $k_{d/2} = m$. Using this basis we have a unitary $V\colon L^2(G,\chi) \to L^2(\mathbf{R}^{d/2}, dt)$, and it maps $\mathscr{S}(G,\chi)$ onto $\mathscr{S}(\mathbf{R}^{d/2})$, see §1.8. Set $\pi = \pi_\chi = \mathrm{ind}_{H\uparrow G}\chi$. The representation π is transformed into the representation π^V having the form

$$\pi^V(s)h(t) = \chi(b(s,t))^{-1} h(s^{-1}t),$$

see §1.8.

In the following we indicate objects defined relative to the subgroup K by a subscript or superscript " 0 ". In $\mathfrak{k}$ we have a Jordan-Hölder basis $\mathscr{B}_0\colon X_1, \ldots, X_{m-1}$. We call the associated Jordan-Hölder sequence $\mathscr{F}_0$. Clearly $\mathfrak{h}(g,\mathscr{F}) = \mathfrak{h}(g_0,\mathscr{F}_0)$, where $g_0 = g|\mathfrak{k}$. We have $\mathfrak{k}_{g_0} = \mathfrak{g}_g \oplus \mathbf{R}X_2$ and writing $J_{g_0} = \{j_1^0 < \cdots < j_{d-2}^0\}$ we have that $j_r^0 = j_{r+1}$, $r = 1, \ldots, d-2$. Moreover, $X_{k_1}, \ldots, X_{k_{d/2-1}}$ is a basis in $\mathfrak{k}$ (mod $\mathfrak{h}(g_0,\mathscr{F}_0)$) of subalgebra type. We set $\pi_0 = \mathrm{ind}_{H\uparrow K}\chi$, and transform π_0 to $\pi_0^{V_0}$ acting on $L^2(\mathbf{R}^{d/2-1}, dt^0)$ by the unitary V_0, just as we did with $\pi = \pi_\chi$, and we get

$$\pi_0^{V_0}(s_0)h_0(t^0) = \chi(b_0(s_0,t^0))^{-1} h_0(s_0^{-1}t^0).$$

We identify $\mathbf{R}^{d/2}$ with $\mathbf{R}^{d/2-1}\times\mathbf{R}$ as follows $t=(t^0, t_{d/2})$. A simple calculation then gives that

$$d\pi^V(X_m)f(t) = -\frac{\partial f}{\partial t_{d/2}}(t),$$
$$d\pi^V(X)f(t) = d\pi_0^{V_0}(e^{-t_{d/2}\operatorname{ad}X_m}X)f(t^0, t_{d/2})$$

for $X\in\mathfrak{k}$, where $d\pi_0^{V_0}(e^{-t_{d/2}\operatorname{ad}X_m}X)$ acts in the first variable.

Let us now note that if $l = sg \in O$, and $y=\sigma_O^{\mathscr{B}}(l)$, then $\langle s\exp uX_2 g, X_j\rangle = \langle sg, X_j\rangle$ for $j<m$ while

$$\begin{aligned}\langle s\exp uX_2 g, X_m\rangle &= \langle sg, e^{-u\operatorname{ad}X_2}X_m\rangle = \langle sg, X_m + u[X_m, X_2]\rangle \\ &= \langle sg, X_m\rangle + u\langle g, [X_m, X_2]\rangle = \langle l, X_m\rangle + u\mu,\end{aligned}$$

where $\mu = \langle g, [X_m, X_2]\rangle$. It follows that $\varphi\circ\sigma_O^{\mathscr{B}^{-1}}(y_1, \dots, y_{d-1}, y_d + u\mu) = \varphi(s\exp uX_2 g) = \varphi(sg) = \varphi\circ\sigma_O^{\mathscr{B}^{-1}}(y_1, \dots, y_{d-1}, y_d)$, since $\varphi\in\mathscr{P}^0_{F(O,\mathscr{F})}(O)$ and $X_2\in\mathfrak{h}(g,\mathscr{F})$. The conclusion is that $\varphi\circ\sigma_O^{\mathscr{B}^{-1}}$ only depends on the variables $y_1, \dots, y_{d-1}$, and therefore $\varphi\circ\sigma_O^{\mathscr{B}^{-1}}(-iX_{j_1}, \dots, -iX_{j_d})\in S(\mathfrak{k}_{\mathbf{C}})$, hence $\Phi_O^{\mathscr{B}}(\varphi)\in U(\mathfrak{k}_{\mathbf{C}})$. It also follows that there exists a function $\overline{\varphi}$ on $\overline{O} = \{l|\mathfrak{k} \mid l\in O\}$ such that $\varphi(l) = \overline{\varphi}(l|\mathfrak{k})$ for all $l\in O$, and $\overline{\varphi}$ is the restriction to $\overline{O}$ of a polynomial function.

We write

$$\varphi\circ\sigma_O^{\mathscr{B}^{-1}}(y_1, \dots, y_d) = \sum_{\alpha=(\alpha_2, \dots, \alpha_{d-1})} a_\alpha(y_1)y_2^{\alpha_2}\cdots y_{d-1}^{\alpha_{d-1}},$$

where $a_\alpha(y_1)$ is a polynomial function in y_1.

Set $O_0 = Kg_0$, and set $\varphi_0 = \overline{\varphi}|O_0$. Then if $l_0\in O_0$ and $l\in O$ with $l|\mathfrak{k} = l_0$ we have $\langle l_0, X_2\rangle = \xi_2 = \langle g, X_2\rangle$. It follows that if $\sigma_{O_0}^{\mathscr{B}_0}(l_0) = (y_2, \dots, y_{d-1})$, then $\sigma_O^{\mathscr{B}}(l) = (\xi_2, y_2, \dots, y_{d-1}, y_d)$, and therefore that

$$\begin{aligned}\varphi\circ\sigma_O^{\mathscr{B}^{-1}}(\xi_2, y_2, \dots, y_{d-1}, y_d) &= \varphi(l) = \varphi_0(l_0) \\ &= \varphi_0\circ\sigma_{O_0}^{\mathscr{B}_0^{-1}}(y_2, \dots, y_{d-1}).\end{aligned}$$

The conclusion is that

$$\varphi_0\circ\sigma_{O_0}^{\mathscr{B}_0^{-1}}(y_2, \dots, y_{d-1}) = \sum_{\alpha=(\alpha_2, \dots, \alpha_{d-1})} a_\alpha(\xi_2)y_2^{\alpha_2}\cdots y_{d-1}^{\alpha_{d-1}}.$$

Now

$$\begin{aligned}\Phi_O^{\mathscr{B}}(\varphi) &= \omega(\varphi\circ\sigma_O^{\mathscr{B}^{-1}}(-iX_{j_1}, \dots, -iX_{j_d})) \\ &= \omega\left(\sum_\alpha a_\alpha(-iX_2)(-iX_{j_2})^{\alpha_2}\cdots(-iX_{j_{d-1}})^{\alpha_{d-1}}\right) \\ &= \sum_\alpha a_\alpha(-iX_2)\omega((-iX_{j_2})^{\alpha_2}\cdots(-iX_{j_{d-1}})^{\alpha_{d-1}}).\end{aligned}$$

Here we have used that X_2 is central in $\mathfrak{k}$. But then

$$\begin{aligned} d\pi_0(\Phi_O^{\mathscr{B}}(\varphi)) &= \sum_\alpha d\pi_0(a_\alpha(-iX_2))d\pi_0(\omega((-iX_{j_2})^{\alpha_2}\cdots(-iX_{j_{d-1}})^{\alpha_{d-1}})) \\ &= \sum_\alpha a_\alpha(\xi_2)d\pi_0(\omega((-iX_{j_1^0})^{\alpha_2}\cdots(-iX_{j_{d-2}^0})^{\alpha_{d-1}})) \\ &= d\pi_0(\omega(\sum_\alpha a_\alpha(\xi_2)(-iX_{j_1^0})^{\alpha_2}\cdots(-iX_{j_{d-2}^0})^{\alpha_{d-1}}) \\ &= d\pi_0(\omega(\varphi_0\circ\sigma_{O_0}^{\mathscr{B}_0^{-1}}(-iX_{j_1^0},\cdots,-iX_{j_{d-2}^0}))) \\ &= d\pi_0(\Phi_{O_0}^{\mathscr{B}_0}(\varphi_0)). \end{aligned}$$

By the induction hypothesis we have that $d\pi_0(\Phi_{O_0}^{\mathscr{B}_0}(\varphi_0)) = \delta_\chi^0(\varphi_0)$, and therefore we get

$$(*) \qquad d\pi_0(\Phi_O^{\mathscr{B}}(\varphi)) = \delta_\chi^0(\varphi_0).$$

We shall now find $\mathrm{Ad}(\exp -tX_m)(\Phi_O^{\mathscr{B}}(\varphi))$ for $t\in\mathbf{R}$. First, setting $X_j^t = \mathrm{Ad}(\exp -tX_m)X_j$, $1\le j\le m$, we get a new Jordan-Hölder basis $\mathscr{B}^t$ compatible with the Jordan-Hölder sequence $\mathscr{F}$. Applying the formula $(*)$ to the Jordan-Hölder basis $\mathscr{B}^t$ we get

$$(**) \qquad d\pi_0(\Phi_O^{\mathscr{B}^t}(\varphi)) = \delta_\chi^0(\varphi_0).$$

Next, define the function φ_t by $\varphi_t(l) = \varphi(\exp tX_m l)$. Clearly $\varphi_t \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$. If $\varphi\circ\sigma_O^{\mathscr{B}^{-1}} = \sum_\alpha c_\alpha y^\alpha$, then

$$\varphi(l) = \sum_\alpha c_\alpha(\langle l, X_{j_1}\rangle)^{\alpha_1}\cdots(\langle l, X_{j_d}\rangle)^{\alpha_d}.$$

and therefore

$$\begin{aligned} \varphi_t(l) = \varphi(\exp tX_m l) &= \sum_\alpha c_\alpha(\langle \exp tX_m l, X_{j_\alpha}\rangle)^\alpha \\ &= \sum_\alpha c_\alpha(\langle l, \mathrm{Ad}(\exp -tX_m)X_{j_1}\rangle)^{\alpha_1}\cdots(\langle l, \mathrm{Ad}(\exp -tX_m)X_{j_d}\rangle)^{\alpha_d} \\ &= \sum_\alpha c_\alpha(\langle l, X_{j_1}^t\rangle)^{\alpha_1}\cdots(\langle l, X_{j_d}^t\rangle)^{\alpha_d}. \end{aligned}$$

Now we see from above that $\varphi_t\circ\sigma_O^{\mathscr{B}^{t^{-1}}} = \varphi\circ\sigma_O^{\mathscr{B}^{-1}}$, so

$$\begin{aligned} \varphi_t\circ\sigma_O^{\mathscr{B}^{t^{-1}}}(-iX_{j_1}^t,\ldots,-iX_{j_d}^t) &= \varphi\circ\sigma_O^{\mathscr{B}^{-1}}(-iX_{j_1}^t,\ldots,-iX_{j_d}^t) \\ &= \mathrm{Ad}(\exp -tX_m)(\varphi\circ\sigma_O^{\mathscr{B}^{-1}}(-iX_{j_1},\ldots,-iX_{j_d})), \end{aligned}$$

which implies that

$$(***) \qquad \Phi_O^{\mathscr{B}^t}(\varphi_t) = \mathrm{Ad}(\exp -tX_m)(\Phi_O^{\mathscr{B}}(\varphi)).$$

Using $(**)$ and $(***)$ we then get

$$d\pi_0(\mathrm{Ad}(\exp -tX_m)\Phi_O^{\mathscr{B}}(\varphi)) = \delta_\chi^0((\varphi_t)_0).$$

But then it follows that

$$\begin{aligned} d\pi^V(\Phi_O^{\mathscr{B}}(\varphi))f(t) &= d\pi_0^{V_0}(\mathrm{Ad}(\exp -t_{d/2}X_m)\Phi_O^{\mathscr{B}}(\varphi))f(t) \\ &= \delta_\chi^{0^{V_0}}((\varphi_{t_{d/2}})_0)f(t^0, t_{d/2}) \\ &= i\varphi_{t_{d/2}}(\exp t_{d/2-1}X_{k_{d/2-1}} \cdots \exp t_1 X_{k_1} g)f(t) \\ &= i\varphi(\exp t_{d/2}X_m \exp t_{d/2-1}X_{k_{d/2-1}} \cdots \exp t_1 X_{k_1} g)f(t) \\ &= \delta_\chi^V(\varphi)f(t) \end{aligned}$$

and therefore

$$d\pi(\Phi_O^{\mathscr{B}}(\varphi)) = \delta_\chi(\varphi).$$

This finishes Case (b), and ends the proof of Theorem 3.1.2.

We have till now only defined $\Phi(\varphi) \in U(\mathfrak{g}_\mathbf{C})$ for $\varphi \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$. We shall now define $\Phi(\varphi) \in U(\mathfrak{g}_\mathbf{C})$ for $\varphi \in \mathscr{P}^1_{F(O,\mathscr{F})}(O)$. To this end, recall from Proposition 2.2.9 that each $\varphi \in \mathscr{P}^1_{F(O,\mathscr{F})}(O)$ can be written uniquely as

$$(\sharp) \qquad \varphi = \sum_{r=1}^{d/2} \varphi_r \psi^{X_{k_r}} + \varphi_0,$$

where $\varphi_0, \varphi_1, \dots, \varphi_{d/2}$ belong to $\mathscr{P}^0_{F(O,\mathscr{F})}(O)$. We set

$$\Phi(\varphi) = -i\sum_{r=1}^{d/2} \Phi(\varphi_r)X_{k_r} + \Phi(\varphi_0).$$

Theorem 3.1.3. *For* $\varphi \in \mathscr{P}^1_{F(O,\mathscr{F})}(O)$ *we have*

$$d\pi_\chi(\Phi(\varphi)) = \delta_\chi(\varphi).$$

Proof. Since δ_χ is $\mathscr{P}^0_F(O)$-linear this follows immediately from Theorem 3.1.2 and the definition of Φ.

We now consider the map ε_χ. Recall (see §2.5) that ε_χ defines an isomorphism

$$\varepsilon_\chi : \mathscr{P}^1_F(O) \to \mathscr{A}^1(G, \chi).$$

We set $\tilde{\Psi}$ to be the map that makes the following diagram commutative:

$$\begin{array}{ccc} \mathscr{P}^1_F(O) & \xrightarrow{\tilde{\Psi}} & J^1_F(O)/I(O) \\ & {\scriptstyle\varepsilon_\chi}\searrow \quad \swarrow{\scriptstyle d\pi} & \\ & \mathscr{A}^1(G,\chi) & \end{array}$$

We shall now construct an explicit map $\Psi : \mathscr{P}_F^1(O) \to \mathscr{A}^1(G, \chi)$ such that Ψ when composed with the quotient map $J_F^1(O) \to J_F^1(O)/I(O)$ is equal to $\tilde{\Psi}$, just as we did for $\tilde{\Phi}$:

We set $\Psi(\varphi) = \Phi(\varphi)$ for $\varphi \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$. For $\varphi \in \mathscr{P}_F^1(O)$ we write φ as in the formula ($\sharp$) above, and set

$$\Psi(\varphi) = -i\frac{1}{2}\sum_{r=1}^{d/2}(\Psi(\varphi_r)X_{k_r} + X_{k_r}\Psi(\varphi_r)) + \Psi(\varphi_0).$$

Theorem 3.1.4. *For* $\varphi \in \mathscr{P}^1_{F(O,\mathscr{F})}(O)$ *we have*

$$d\pi_\chi(\Psi(\varphi)) = \varepsilon_\chi(\varphi).$$

Proof. We have

$$\begin{aligned} d\pi_\chi(\Psi(\varphi)) &= -i\frac{1}{2}\sum_{r=1}^{d/2}(d\pi(\Psi(\varphi_r))d\pi(X_{k_r}) + d\pi(X_{k_r})d\pi(\Psi(\varphi_r))) + d\pi(\Psi(\varphi_0)) \\ &= -i\frac{1}{2}\sum_{r=1}^{d/2}(\delta_\chi(\varphi_r)d\pi(X_{k_r}) + d\pi(X_{k_r})\delta_\chi(\varphi_r)) + \delta_\chi(\varphi_0). \end{aligned}$$

Now

$$\delta_\chi(\varphi) = -i\sum_{r=1}^{d/2}\delta_\chi(\varphi_r)d\pi(X_{k_r}) + \delta_\chi(\varphi_0),$$

hence

$$\delta_\chi(\overline{\varphi})^* = i\sum_{r=1}^{d/2}d\pi(X_{k_r})\delta_\chi(\varphi_r) - \delta_\chi(\varphi_0).$$

We see that the difference of the last two expressions is equal to twice the result of the first calculation, hence

$$\varepsilon_\chi(\varphi) = \tfrac{1}{2}(\delta_\chi(\varphi) - \delta_\chi(\overline{\varphi})^*) = d\pi(\Psi(\varphi)).$$

This ends the proof of the theorem.

3.2. Construction of unitary representations with an explicit cross-section for the differential. Suppose now that $(p_1, \dots, p_{d/2}, q_1, \dots, q_{d/2})$ are global canonical coordinates on O such that $p_1, \dots, p_{d/2} \in \mathscr{P}^1_{F(O,\mathscr{F})}(O)$ and $q_1, \dots, q_{d/2} \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$. As in §2.7 we write for each $X \in \mathfrak{g}$

$$\psi^X = \sum_{r=1}^{d/2} a_{X,r}(q)p_r + a_{X,0}(q),$$

where $a_{X,r} \in \mathscr{P}(\mathbf{R}^{d/2})$, $r = 0, 1, \dots, d/2$.

We set $u_r = \Psi(p_r)$, $v_r = \Psi(q_r)$, $r = 1, \dots, d/2$. We can then formulate:

Theorem 3.2.1 There exists one and only one strongly continuous, unitary representation π of G on $L^2(\mathbf{R}^{d/2})$ such that $\mathscr{D}(\mathbf{R}^{d/2}) \subset H_\pi^\infty$, and such that

$$d\pi(X)f(t) = \sum_{r=1}^{d/2} a_{X,r}(t)\frac{\partial f}{\partial t_r}(t) + ia_{X,0}(t)f(t) + \frac{1}{2}\sum_{r=1}^{d/2} \frac{\partial a_{X,r}}{\partial t_r}(t)f(t)$$

for $f \in \mathscr{D}(\mathbf{R}^{d/2})$. The representation π is in the equivalence class of irreducible representations associated with the orbit O, and we have

$$d\pi(u_r) = D_r, \quad d\pi(v_r) = iM_r,$$

where $D_r f = \partial f/\partial t_r$ and $M_r f(t) = t_r f(t)$, $r = 1, \dots, d/2$.

Proof. This follows from Theorem 3.1.4 and [9, Theorem 3.2.3] (and also invoking a few details from proof of this theorem).

4. Coarse and fine $\mathscr{F}$-stratification of the dual of a nilpotent Lie algebra

The main purpose of this section is to define the "fine $\mathscr{F}$-stratification"of the dual of a nilpotent Lie algebra with respect to a given Jordan-Hölder sequence $\mathscr{F}$. This will be used in an essential way in §5. The fine $\mathscr{F}$-stratification is a refinement of what we call the "coarse $\mathscr{F}$-stratification", which is due to Pukanszky.

4.1. Jump indices for skewsymmetric bilinear form with respect to flag. We start by defining the "jump indices"of a skewsymmetric bilinear form.

Let V be a finite dimensional vector space over the field $\mathbf{K}$, and let $B: V \times V \to \mathbf{K}$ be a skewsymmetric bilinear form on V. If M is any subset we denote by $M^\perp$ the set $M^\perp = \{v \in V \mid B(u,v) = 0 \text{ for all } u \in M\}$. We define the radical V_B of B to be the subspace $V^\perp$.

Suppose that

$$\mathscr{F}: V = V_m \supset V_{m-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}$$

is a flag of subspaces, i.e. an increasing sequence of subspaces such that $\dim V_j = j$.

Set

$$J_B = \{1 \le j \le m \mid V_j \not\subset V_{j-1} + V_B\}.$$

We call J_B the set of jump indices of B relative to the given flag $\mathscr{F}$.

Let $x_j \in V_j \setminus V_{j-1}$, $j = 1, \dots, m$. Then $x_1, \dots, x_m$ is a basis in V. We have

$$J_B = \{1 \le j \le m \mid x_j \notin V_{j-1} + V_B\}.$$

If J_B is nonempty we can write $J_B = \{j_1 < \cdots < j_d\}$. Then $x_{j_1}, \dots, x_{j_d}$ is a basis for $V \pmod{V_B}$. This implies that the matrix

$$M_B = [B(x_{j_u}, x_{j_v})]_{1 \le u,v \le d}$$

is nonsingular.

Set B_j to be the restriction of B to $V_j \times V_j$, $j = 1, \dots, m$. The symbol $(V_j)_{B_j}$ then denotes the radical of B_j. For the set of jump indices J_{B_j} of B_j relative to the flag

$$\mathscr{F}_j : V_j \supset V_{j-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}$$

we have

$$J_{B_j} = \{1 \le k \le j \mid x_k \notin V_{k-1} + (V_j)_{B_j}\}.$$

Let us note that $J_{B_j} \subset J_B$; in fact, we clearly have that $J_B = \{1 \le j \le m \mid x_j \notin V_{j-1} + V_B \cap V_j\}$, and also $V_B \cap V_j \subset (V_j)_{B_j}$; it follows that $x_k \in V_{k-1} + V_B \cap V_j \Rightarrow x_k \in V_{k-1} + (V_j)_{B_j}$, and therefore $x_k \notin V_{k-1} + (V_j)_{B_j} \Rightarrow x_k \notin V_{k-1} + V_B \cap V_j$. This implies that $J_{B_j} \subset J_B$. In general we have

$$\varnothing = J_{B_1} \subset J_{B_2} \subset \cdots \subset J_{B_{m-1}} \subset J_{B_m} = J_B.$$

Set

$$K_B = \{1 \le j \le m \mid J_{B_{j-1}} \ne J_{B_j}\}.$$

Now either

$$\dim V_j/(V_j)_{B_j} = \dim V_{j-1}/(V_{j-1})_{B_{j-1}}$$

or

$$\dim V_j/(V_j)_{B_j} = \dim V_{j-1}/(V_{j-1})_{B_{j-1}} + 2$$

[3, 1.12.2]. It follows that either $\operatorname{card}(J_{B_j}) = \operatorname{card}(J_{B_{j-1}})$ or $\operatorname{card}(J_{B_j}) = \operatorname{card}(J_{B_{j-1}}) + 2$. Since $J_{B_1} = \varnothing$ and $\operatorname{card}(J_{B_m}) = \operatorname{card}(J_B) = d$ is even we get that $\operatorname{card}(K_B) = d/2$.

Another characterization of K_B is

$$(*) \qquad K_B = \{1 \le j \le m \mid j \in J_{B_j}\};$$

in fact, if $j \in J_{B_j}$, then $j \notin J_{B_{j-1}}$, so $j \in K_B$. Conversely, suppose that $j \notin J_{B_j}$. Then $x_j \in V_{j-1} + (V_j)_{B_j}$ and therefore $(V_{B_j}) \cap V_{j-1} = (V_{j-1})_{B_{j-1}}$, hence $J_{B_{j-1}} = J_{B_j}$. This proves $(*)$. In particular we get $j \in K_B \Rightarrow j \in J_{B_j} \subset J_B$, so $K_B \subset J_B$.

Still another characterization of K_B is as follows: We know that $P(B, \mathscr{F}) = \sum_{j=1}^m (V_j)_{B_j}$ is a maximal isotropic subspace in V with respect to B, and that $P(B, \mathscr{F}) \cap V_j = P(B_j, \mathscr{F}_j)$ [3, 1.12.3]. From this we see immediately that

$$K_B = \{1 \le j \le m \mid V_j \not\subset V_{j-1} + P(B, \mathscr{F})\}.$$

We set

$$\mathscr{J}_B = (J_{B_1}, J_{B_2}, \dots, J_{B_{m-1}}, J_{B_m}).$$

Definition 4.1.1. *We call $\mathscr{J}_B$ for the m-tuple of jump indices for B with respect to the given flag $\mathscr{F}$.*

We write

$$J_B = \{j_1 < \cdots < j_d\}, \quad K_B = \{k_1 < \cdots < k_{d/2}\}.$$

If $j < k_1$ we have $J_{B_j} = \varnothing$ and if $j > k_{d/2}$ we have $J_{B_j} = J_B$. For $k_r \le j < k_{r+1}$ we have $J_{B_{k_r}} = J_{B_j} \neq J_{B_{k_{r+1}}}$.

We define the finite sequence $l_1, \dots, l_{d/2}$ in $\{1, \dots, m\}$ recursively as follows

$$J_{B_{k_1}} = \{l_1 < k_1\}, \quad J_{B_{k_{r+1}}} = J_{B_{k_r}} \cup \{l_{r+1} < k_{r+1}\}, \quad r < d/2 - 1,$$

and we set $L_B = \{l_1, \dots, l_{d/2}\}$. Then $J_B = K_B \cup L_B$ as a disjoint union. If $k_r \le j < k_{r+1}$ we have that $K_{B_j} = \{k_1 < \cdots < k_r\}$ and $L_{B_j} = \{l_1, \dots, l_r\}$.

We have that $x_{k_1}, \dots, x_{k_{d/2}}, x_{l_1}, \dots, x_{l_{d/2}}$ is a basis in $V_j \pmod{V_{B_j}}$ for all $j \ge k_{d/2}$. If $k_r \le j < k_{r+1}$, $r < d/2$, we have that $x_{k_1}, \dots, x_{k_r}, x_{l_1}, \dots, x_{l_r}$ is a basis in $V_j \pmod{(V_j)_{B_j}}$.

4.2. Coarse and fine $\mathscr{F}$-stratification of the dual of a nilpotent Lie algebra. Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, and let

$$\mathscr{F}: \ \mathfrak{g} = \mathfrak{g}_m \supset \mathfrak{g}_{m-1} \supset \cdots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0 = \{0\}$$

be a fixed Jordan-Hölder sequence in $\mathfrak{g}$, and let $\mathscr{B}: X_1, \dots, X_m$ be a compatible Jordan-Hölder basis in $\mathfrak{g}$, i.e. $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$.

For each $g \in \mathfrak{g}^*$ we have defined the skew-symmetric bilinear form $B_g: \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$. We set J_g to be the set of jump indices of B_g relative to the flag $\mathscr{F}$, i.e. $J_g = J_{B_g}$ with the notation from §4.1. The restriction $(B_g)_j$ of B_g to $\mathfrak{g}_j \times \mathfrak{g}_j$ is equal to B_{g_j}, where $g_j = g|\mathfrak{g}_j$. Therefore, if we set $J_g^j = J_{B_{g_j}}$ we have that the m-tuple of jump indices of B_g is

$$\mathscr{J}_g = \mathscr{J}_{B_g} = (J_g^1, J_g^2, \dots, J_g^{m-1}, J_g^m).$$

Let us note that we have

$$J_g^j = \{1 \le k \le j \mid X_k \notin \mathfrak{g}_{k-1} + (\mathfrak{g}_j)_{g_j}\}.$$

We set $\mathscr{E} = \{J_g \mid g \in \mathfrak{g}^*\}$, and for $e \in \mathscr{E}$ we set $\Omega_e = \{g \in \mathfrak{g}^* \mid J_g = e\}$. Similarly we define $\mathscr{E}_j = \{J_g^j \mid g \in \mathfrak{g}^*\}$, and we set $\boldsymbol{\mathscr{E}} = \{\mathscr{J}_g \mid g \in \mathfrak{g}^*\}$. Clearly $\boldsymbol{\mathscr{E}} \subset \mathscr{E}_1 \times \cdots \times \mathscr{E}_m$.

If $\varepsilon = (e_1, \dots, e_m) \in \boldsymbol{\mathscr{E}}$ we have that $e_1 \subset e_2 \subset \cdots \subset e_{m-1} \subset e_m$, and we define

$$K_\varepsilon = \{1 \le j \le m \mid e_{j-1} \not\subset e_j\} = \{1 \le j \le m \mid j \in e_j\}.$$

We write $K_\varepsilon = \{k_1 < \cdots < k_{d/2}\}$, and define the sequence $l_1, \dots, l_{d/2}$ recursively as follows:

$$e_{k_1} = \{l_1 < k_1\}, \quad e_{k_{r+1}} = e_{k_r} \cup \{l_{r+1} < k_{r+1}\}, \quad r < d/2 - 1,$$

and we set $L_\varepsilon = \{l_1, \dots, l_{d/2}\}$. Then $e_m = K_\varepsilon \cup L_\varepsilon$ as a disjoint union. We set $\varepsilon_j = (e_1, \dots, e_j)$. If $k_r \le j < k_{r+1}$ we have that $K_{\varepsilon_j} = \{k_1 < \dots < k_r\}$ and $L_{\varepsilon_j} = \{l_1, \dots, l_r\}$, and $e_j = K_{\varepsilon_j} \cup L_{\varepsilon_j}$.

Let e be a subset of $\{1, \dots, m\}$. If $e = \emptyset$ we set $\mathbf{M}_e = 1$, $Q_e = 1$ and $P_e = 1$. If $e \neq 0$ we write $e = \{j_1 < \dots < j_d\}$ and set for each $g \in \mathfrak{g}^*$ $\mathbf{M}_e(g)$ to be the $d \times d$-matrix defined by

$$\mathbf{M}_e(g) = [\langle g, [X_{j_u}, X_{j_v}]\rangle]_{1 \le u, v \le d},$$

and set

$$Q_e(g) = \det \mathbf{M}_e(g),$$

and

$$P_e(g) = \operatorname{Pf} \mathbf{M}_e(g).$$

For $\varepsilon \in \boldsymbol{\mathscr{E}}$ we set

$$\Omega_\varepsilon = \{g \in \mathfrak{g}^* \mid \mathscr{J}_g = \varepsilon\}.$$

We clearly have that $\mathfrak{g}^* = \bigcup_{\varepsilon \in \boldsymbol{\mathscr{E}}} \Omega_\varepsilon$ as a finite, disjoint union. We call this partition of $\mathfrak{g}^*$ the *fine $\mathscr{F}$-stratification of* $\mathfrak{g}^*$ with respect to $\mathscr{F}$.

Similarly we call the partition $\mathfrak{g}^* = \bigcup_{e \in \mathscr{E}} \Omega_e$ into finitely many disjoint sets for the *coarse $\mathscr{F}$-stratification of* $\mathfrak{g}^*$ with respect to $\mathscr{F}$.

Let $\varepsilon \in \boldsymbol{\mathscr{E}}$ and write $\varepsilon = (e_1, \dots, e_m)$. We have

Lemma 4.2.1. *The polynomial functions P_{e_j} and Q_{e_j} are constant on each coadjoint orbit O contained in Ω_ε.*

Proof. If $O \subset \Omega_\varepsilon$ and if $g \in O$ then $J_g^j = e_j$. Let us write $e_j = \{j_1 < \dots < j_d\}$. Since $\mathfrak{g}_j$ is an ideal we can write for each $s \in G$ and each $1 \le v \le d$:

$$\operatorname{Ad}(s^{-1}) X_{j_v} = \sum_{u=1}^{d} a_{uv} X_{j_u} + Z_v,$$

where $Z_u \in (\mathfrak{g}_j)_{g_j}$, and we have that $a_{uv} = 0$ for $u > v$ and $a_{vv} = 1$. If we let $\mathbf{A}$ be the matrix $[a_{uv}]_{1 \le u, v \le d}$ we have

$$\begin{aligned} \langle sg, [X_{j_u}, X_{j_v}]\rangle &= \langle g, [\operatorname{Ad}(s^{-1})X_{j_u}, \operatorname{Ad}(s^{-1})X_{j_v}]\rangle \\ &= \sum_{p,q=1}^{d} a_{pu} \langle g, [X_{j_p}, X_{j_q}]\rangle a_{qv} = ({}^t\mathbf{A}\mathbf{M}_{e_j}(g)\mathbf{A})_{uv}, \end{aligned}$$

which means that $\mathbf{M}_{e_j}(sg) = {}^t\mathbf{A}\mathbf{M}_{e_j}(g)\mathbf{A}$. Now since $\mathbf{A}$ is an upper triangular matrix whose diagonal elements are all equal to 1 we have that $\det \mathbf{A} = 1$, hence $P_{e_j}(sg) = \operatorname{Pf} \mathbf{M}_{e_j}(sg) = \operatorname{Pf} {}^t\mathbf{A}\mathbf{M}_{e_j}(g)\mathbf{A} = \det \mathbf{A} \operatorname{Pf} \mathbf{M}_{e_j} = \operatorname{Pf} \mathbf{M}_{e_j} = P_{e_j}(g)$. This ends the proof of the lemma, since $Q_{e_j} = P_{e_j}^2$.

Recall (see §3.1) that we have set

$$\mathfrak{h}(g, \mathscr{F}) = \sum_{j=1}^{m} (\mathfrak{g}_j)_{g_j},$$

and that $\mathfrak{h}(g,\mathcal{F})$ is a polarization at g [3, 1.2.10], the *Vergne polarization* at g associated with $\mathcal{F}$.

If $g \in \Omega_\varepsilon$ we also write $K_g = K_\varepsilon$ and $L_g = L_\varepsilon$. For $g \in \mathfrak{g}^*$ we can characterize K_g as follows:

$$K_g = \{1 \le j \le m \mid X_j \notin \mathfrak{g}_{j-1} + \mathfrak{h}(g,\mathcal{F})\}.$$

5. Construction of canonical coordinates

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. In this section we shall show how one can explicitly construct global canonical coordinates on each coadjoint orbit of G. Actually, we do more than that: We give an algorithm for constructing the rational functions $p_1^\varepsilon, \dots, p_{d/2}^\varepsilon, q_1^\varepsilon, \dots, q_{d/2}^\varepsilon$ appearing in the theorem below. This theorem is the main result of this section. As a by-product we obtain non-inductive proofs of two basic facts from the Kirillov theory (5.4.6), namely that one can associate a well-defined class of irreducible representations of G with each coadjoint orbit O of G, and that the image of the universal enveloping algebra by the differential of such a representation can be identified with the algebra of all polynomial differential operators on some space $\mathbf{R}^n$ (the Weyl algebra).

5.1. Formulation of theorem. Let $\mathcal{F}$ be a Jordan-Hölder sequence in $\mathfrak{g}$, and let $\mathcal{B}$: $X_1, \dots, X_m$ be a basis in $\mathfrak{g}$ compatible with $\mathcal{F}$. Recall that we have defined the *fine $\mathcal{F}$-stratification* $\mathfrak{g}^* = \bigcup_{\varepsilon \in \mathcal{E}} \Omega_\varepsilon$ of $\mathfrak{g}^*$, and that $F(O,\mathcal{F})$ is the G-invariant polarization of O defined by the *Vergne polarization* $\mathfrak{h}(g) = \mathfrak{h}(g,\mathcal{F})$ at $g \in \mathfrak{g}^*$ (§4.2).

Theorem 5.1.1. *There exists for each $\varepsilon \in \mathcal{E}$ rational functions $p_1^\varepsilon, \dots, p_{d/2}^\varepsilon$, $q_1^\varepsilon, \dots, q_{d/2}^\varepsilon$ on $\mathfrak{g}^*$ with the following properties: For each orbit O contained in Ω_ε the restrictions $q_1^O, \dots, q_{d/2}^O$ of $q_1^\varepsilon, \dots, q_{d/2}^\varepsilon$ to O are in $\mathcal{P}^0_{F(O,\mathcal{F})}(O)$ and the restrictions $p_1^O, \dots, p_{d/2}^O$ of $p_1^\varepsilon, \dots, p_{d/2}^\varepsilon$ to O are in $\mathcal{P}^1_{F(O,\mathcal{F})}(O)$, and $p_1^O, \dots, p_{d/2}^O, q_1^O, \dots, q_{d/2}^O$ are canonical coordinates on O.*

We shall prove Theorem 5.1.1 in §§5.2, 5.3, 5.4 and 5.5. In the course of our proof we shall describe an algorithm for calculating the the rational functions $p_1^\varepsilon, \dots, p_{d/2}^\varepsilon, q_1^\varepsilon, \dots, q_{d/2}^\varepsilon$ appearing in the theorem, and, in fact, this algorithm produces rational functions with denominators from the algebra generated by $P_{e_1}, \dots, P_{e_m}$.

5.2. The rational functions S_j^ε; defining q_u^ε. The main goal of this section is to define the rational functions $q_1^\varepsilon, \dots, q_{d/2}^\varepsilon$ appearing in Theorem 5.1.1.

For $g \in \Omega_\varepsilon$ we define the $(m \times d/2)$-matrix $\boldsymbol{\tau}^\varepsilon(g) = \left[\tau^\varepsilon_{j\,u}(g)\right]_{\substack{1\le j\le m\\ 1\le u\le d/2}}$ as follows:

$$X_j = \sum_{u=1}^{d/2} \tau^\varepsilon_{j\,u}(g) X_{k_u} + B^\varepsilon_j(g),$$

where $B^\varepsilon_j(g) \in \mathfrak{h}(g)$. The matrix $\boldsymbol{\tau}^\varepsilon(g)$ is well defined since $X_{k_1}, \dots, X_{k_{d/2}}$ is a basis for $\mathfrak{g}$ (mod $\mathfrak{h}(g)$) for $g \in \Omega_\varepsilon$.

Lemma 5.2.1. *For all* $g \in \Omega_\varepsilon$ *and* $h \in H = H(g) = \exp \mathfrak{h}(g)$ *we have that* $\boldsymbol{\tau}^\varepsilon(hg) = \boldsymbol{\tau}^\varepsilon(g)$ *and* $B_j(hg) = B_j(g)$, $1 \le j \le m$.

Proof. In general we have $\mathfrak{h}(sg) = \mathrm{Ad}(s)\mathfrak{h}(g)$, and since $\mathfrak{h}(g)$ is a subalgebra we have $\mathfrak{h}(hg) = \mathrm{Ad}(h)\mathfrak{h}(g) = \mathfrak{h}(g)$ for $h \in H(g)$. Therefore writing

$$X_j = \sum_{u=1}^{d/2} \tau^\varepsilon_{j\,u}(hg) X_{k_u} + B_j(hg),$$

where $B_j(hg) \in \mathfrak{h}(hg) = \mathfrak{h}(g)$ we see that $B_j(hg) = B_j(g)$, and $\tau^\varepsilon_{j\,u}(hg) = \tau^\varepsilon_{j\,u}(g)$ for all $g \in \Omega_\varepsilon$, $h \in H(g)$.

For $1 \le j \le m$ we define the function S^ε_j on Ω_ε as follows

$$S^\varepsilon_j(g) = \langle g, B^\varepsilon_j(g)\rangle; \tag{$*$}$$

we also have

$$S^\varepsilon_j(g) = \langle g, X_j\rangle - \sum_{u=1}^{d/2} \tau^\varepsilon_{j\,u}(g)\langle g, X_{k_u}\rangle \tag{$**$}$$

Lemma 5.2.2. *For* $g \in \Omega_\varepsilon$ *and* $h \in H = H(g) = \exp \mathfrak{h}(g)$ *we have* $S^\varepsilon_j(hg) = S^\varepsilon_j(g)$.

Proof. For $h \in H(g)$ we have $S^\varepsilon_j(hg) = \langle hg, B_j(hg)\rangle = \langle hg, B_j(g)\rangle$ Now $h = \exp X$, $X \in \mathfrak{h}(g)$, hence $\langle hg, B_j(g)\rangle = \langle g, e^{-\mathrm{ad} X} B_j(g)\rangle = \langle g, B_j(g)\rangle$, since $\mathfrak{h}(g)$ is an isotropic subalgebra.

It follows from Lemma 5.2.2 that S^ε_j when restricted to an orbit $O = Gg$ contained in Ω_ε is constant on the fibers of the fibration defined by $H = H(g)$. The definition of $q^\varepsilon_1, \dots, q^\varepsilon_{d/2}$ on Ω_ε now are as follows: Write $L_g = \{\bar{l}_1 < \cdots < \bar{l}_{d/2}\}$ and set $q^\varepsilon_u(g) = S^\varepsilon_{\bar{l}_u}(g)$ for $g \in \Omega_\varepsilon$, $u = 1, \dots, d/2$. In order to be able to extend $q^\varepsilon_1, \dots, q^\varepsilon_{d/2}$ to all of $\mathfrak{g}^*$ as rational functions and to give an algorithm for computing such extensions, we shall have to explore the matrix $\boldsymbol{\tau}^\varepsilon$ in greater detail.

5.3. The matrix $\boldsymbol{\tau}$. Let $\varepsilon \in \mathscr{E}$, and let d be the dimension of orbits contained in Ω_ε.

For each $\mathfrak{g} \in \Omega_\varepsilon$ we define the $(m \times d)$-matrix $\boldsymbol{\rho}^{d/2}(g)$ as follows: For each $1 \leq j \leq m$ we write

$$X_j = \sum_{u=1}^{d/2} \rho^{d/2}_{j\,2u-1}(g) X_{k_u} + \sum_{u=1}^{d/2} \rho^{d/2}_{j\,2u} X_{l_u} + A^{d/2}_j(g),$$

where $A^{d/2}_j(g) \in \mathfrak{g}_g$.

For each $1 \leq v \leq d/2$ we then have

$$(\#) \quad \langle g, [X_j, X_{k_v}]\rangle = \sum_{u=1}^{d/2} \rho^{d/2}_{j\,2u-1}(g)\langle g, [X_{k_u}, X_{k_v}]\rangle + \sum_{u=1}^{d/2} \rho^{d/2}_{j\,2u}(g)\langle g, [X_{l_u}, X_{k_v}]\rangle,$$

$$\langle g, [X_j, X_{l_v}]\rangle = \sum_{u=1}^{d/2} \rho^{d/2}_{j\,2u-1}(g)\langle g, [X_{k_u}, X_{l_v}]\rangle + \sum_{u=1}^{d/2} \rho^{d/2}_{j\,2u}(g)\langle g, [X_{l_u}, X_{l_v}]\rangle.$$

In order to rewrite the relations ($\sharp$) in a more compact form we introduce for any $g \in \mathfrak{g}^*$ the $(d \times d)$-matrix $\boldsymbol{\mu}^{d/2}$ and the $(m \times d/2)$-matrix $\boldsymbol{\sigma}^{d/2}(g)$ as follows

$$\sigma^{d/2}_{j\,2u-1} = \langle g, [X_j, X_{k_u}]\rangle, \quad 1 \leq j \leq m, \ 1 \leq u \leq d/2,$$
$$\sigma^{d/2}_{j\,2u} = \langle g, [X_j, X_{l_u}]\rangle, \quad 1 \leq j \leq m, \ 1 \leq u \leq d/2,$$

and

$$\mu^{d/2}_{2u-1\,2v-1}(g) = \langle g, [X_{k_u}, X_{k_v}]\rangle, \quad 1 \leq u, v \leq d/2,$$
$$\mu^{d/2}_{2u-1\,2v}(g) = \langle g, [X_{k_u}, X_{l_v}]\rangle, \quad 1 \leq u, v \leq d/2,$$
$$\mu^{d/2}_{2u\,2v-1}(g) = \langle g, [X_{l_u}, X_{k_v}]\rangle, \quad 1 \leq u, v \leq d/2,$$
$$\mu^{d/2}_{2u\,2v}(g) = \langle g, [X_{l_u}, X_{l_v}]\rangle, \quad 1 \leq u, v \leq d/2.$$

The relations ($\sharp$) can then be expressed as

$$\boldsymbol{\sigma}^{d/2}(g) = \boldsymbol{\rho}^{d/2}(g)\boldsymbol{\mu}^{d/2}(g),$$

for $g \in \Omega_\varepsilon$, hence we have for $g \in \Omega_\varepsilon$:

$$\boldsymbol{\rho}^{d/2}(g) = \boldsymbol{\sigma}^{d/2}(g)\boldsymbol{\mu}^{d/2}(g)^{-1}.$$

We shall take this formula as the definition of $\boldsymbol{\rho}^{d/2}(g)$ when g does not belong to Ω_ε. In this way the entries of the matrix $\boldsymbol{\rho}^{d/2}(g)$ are rational functions on $\mathfrak{g}^*$ with denominators equal to $P_{d/2}(g) = \mathrm{Pf}\,\boldsymbol{\mu}^{d/2}(g)$.

Similarly we define for $1 \leq r < d/2$ the $(k_{r+1} - 1) \times 2r$-matrix $\boldsymbol{\rho}^r(g)$ as follows: For each $1 \leq j \leq (k_{r+1} - 1)$ we write

$$X_j = \sum_{u=1}^{r} \rho^r_{j\,2u-1}(g) X_{k_u} + \sum_{u=1}^{r} \rho^r_{j\,2u}(g) X_{l_u} + A^r_j(g),$$

where $A^r_j(g) \in (\mathfrak{g}_k)_{g_k}$, where $k = k_{r+1} - 1$.

We also define the $(k_{r+1}-1)\times 2r$-matrix $\boldsymbol{\sigma}^r(g)$ and the $(2r\times 2r)$-matrix $\boldsymbol{\mu}^r(g)$ as follows

$$\sigma^r_{j\,2u-1}(g)=\langle g\,,[X_j\,,X_{k_u}]\rangle\,,\quad 1\le j<k_{r+1}\,,\ 1\le u\le r\,,$$
$$\sigma^r_{j\,2u}(g)=\langle g\,,[X_j\,,X_{l_u}]\rangle\,,\quad 1\le j<k_{r+1}\,,\ 1\le u\le r\,,$$

and

$$\mu^r_{2u-1\,2v-1}(g)=\langle g\,,[X_{k_u}\,,X_{k_v}]\rangle\,,\quad 1\le u\,,v\le r\,,$$
$$\mu^r_{2u-1\,2v}(g)=\langle g\,,[X_{k_u}\,,X_{l_v}]\rangle\,,\quad 1\le u\,,v\le r\,,$$
$$\mu^r_{2u\,2v-1}(g)=\langle g\,,[X_{l_u}\,,X_{k_v}]\rangle\,,\quad 1\le u\,,v\le r\,,$$
$$\mu^r_{2u\,2v}(g)=\langle g\,,[X_{l_u}\,,X_{l_v}]\rangle\,,\quad 1\le u\,,v\le r\,,$$

and in this way we have as above

$$\boldsymbol{\sigma}^r(g)=\boldsymbol{\rho}^r(g)\boldsymbol{\mu}^r(g)$$

and

$$\boldsymbol{\rho}^r(g)=\boldsymbol{\sigma}^r(g)\boldsymbol{\mu}^r(g)^{-1}$$

for $g\in\Omega_\varepsilon$. We take the last relation as the definition of $\boldsymbol{\rho}^r(g)$ when g is not contained in Ω_ε. In this way the entries of $\boldsymbol{\rho}^r(g)$ are rational functions in g with denominators in the algebra generated by the polynomials by $P_r(g)=\mathrm{Pf}\boldsymbol{\mu}^r(g)$, $1\le r\le d/2$. Note that the matrices $\boldsymbol{\sigma}^r$ and $\boldsymbol{\mu}^r$, $r=1\,,\dots\,,d/2$, are submatrices of $\boldsymbol{\sigma}^{d/2}$ and $\boldsymbol{\mu}^{d/2}$, respectively.

We note that $\boldsymbol{\rho}^r(g)$ depends only on $g|\mathfrak{g}_{k_r}$. We write this symbolically as $\boldsymbol{\rho}^r(g)=\boldsymbol{\rho}^r(\xi_1\,,\dots\,,\xi_{k_r})$, where $(\xi_1\,,\dots\,,\xi_m)$ are the coordinates of g with respect to the basis of $\mathfrak{g}^*$ dual to $X_1\,,\dots\,,X_m$.

For $g\in\Omega_\varepsilon$ we have defined the $(m\times d/2)$-matrix $\boldsymbol{\tau}(g)=\left[\tau^\varepsilon_{j\,u}\right]_{\substack{1\le j\le m\\ 1\le u\le d/2}}$ as follows:

$$X_j=\sum_{u=1}^{d/2}\tau^\varepsilon_{j\,u}(g)X_{k_u}+B_j(g)\,,$$

where $B_j(g)\in\mathfrak{h}(g)$.

We shall give a recursive way of obtaining $\tau^\varepsilon_{j\,u}$:

If $j<k_1$ we have that $\tau^\varepsilon_{j\,u}(g)=0$ for $1\le u\le d/2$.

Suppose then that $\tau^\varepsilon_{j\,u}(g)$ has been found for all $j<k_r$, $1\le u\le d/2$. If $r<d$ we shall determine $\tau^\varepsilon_{j\,u}$ for all $k_r\le j<k_{r+1}$, $1\le u\le d/2$, and if $r=d$ we shall determine $\tau^\varepsilon_{j\,u}$ for all $k_{d/2}\le j\le m$, $1\le u\le d/2$. So suppose $k_r\le j<k_{r+1}$ (resp. $k_d\le j\le m$). We can then write

$$X_j=\sum_{u=1}^{r}\rho^r_{j\,2u-1}(g)X_{k_u}+\sum_{u=1}^{r}\rho^r_{j\,2u}(g)X_{l_u}+A^r_j(g).$$

We also write

$$X_{l_u} = \sum_{v=1}^{d/2} \tau_{l_u v}(g) X_{k_v} + B_{l_u}(g),$$

$u = 1, \dots, d$.

Now if $u \leq r$, then $l_u < k_u < k_r$, so

$$X_{l_u} = \sum_{v=1}^{r} \tau^{\varepsilon}_{l_u v}(g) X_{k_v} + B_{l_u}(g),$$

$u = 1, \dots, r$, hence

$$\begin{aligned}
X_j &= \sum_{u=1}^{r} \rho^{r}_{j\,2u-1}(g) X_{k_u} + \sum_{u=1}^{r} \rho^{r}_{j\,2u}(g) \left(\sum_{v=1}^{r} \tau^{\varepsilon}_{l_u v}(g) X_{k_v} + B^{\varepsilon}_{l_u}(g) \right) + A^{r}_{j}(g) \\
&= \sum_{u=1}^{r} \rho^{r}_{j\,2u-1}(g) X_{k_u} + \sum_{u=1}^{r} \sum_{v=1}^{r} \rho^{r}_{j\,2u}(g) \tau^{\varepsilon}_{l_u v}(g) X_{k_v} \\
&\quad + \sum_{u=1}^{r} \rho^{r}_{j\,2u}(g) B^{\varepsilon}_{l_u}(g) + A^{r}_{j}(g) \\
&= \sum_{u=1}^{r} (\rho^{r}_{j\,2u-1}(g) + \sum_{v=1}^{r} \rho^{r}_{j\,2v}(g) \tau^{\varepsilon}_{l_v u}(g)) X_{k_u} \\
&\quad + \sum_{u=1}^{r} \rho^{r}_{j\,2u}(g) B^{\varepsilon}_{l_u}(g) + A^{r}_{j}(g).
\end{aligned}$$

Now since the last term is clearly contained in $\mathfrak{h}(g)$ we have for $k_r \leq j < k_{r+1}$ $(k_d \leq j \leq m)$:

$$\begin{aligned}
&\tau^{\varepsilon}_{j\,u}(g) = \rho^{r}_{j\,2u-1}(g) + \sum_{v=1}^{r} \rho^{r}_{j\,2v}(g) \tau^{\varepsilon}_{l_v u}(g), \quad 1 \leq u \leq r, \\
&\tau^{\varepsilon}_{j\,u}(g) = 0, \quad r < u \leq d/2.
\end{aligned}$$

We take this as the defining relations for $\boldsymbol{\tau}^{\varepsilon}(g)$ when g does not belong to Ω_{ε}. In this way $\tau^{\varepsilon}_{j\,u}$ is a rational function on $\mathfrak{g}^*$ with denominators from the algebra genarated by the polynomials $P_1, \dots, P_{d/2}$. We see that $\tau^{\varepsilon}_{j\,u}(g) = \tau^{\varepsilon}_{j\,u}(\xi_1, \dots, \xi_{k_r})$ for $k_r \leq j < k_{r+1}$.

Lemma 5.3.1. *The functions $\tau^{\varepsilon}_{j\,u}$, when restricted to a single orbit O contained in Ω_{ε}, belong to $\mathscr{P}^{0}_{F(O,\mathscr{F})}(O)$.*

Proof. We already know that $\tau^{\varepsilon}_{j\,u}$ are rational functions with denominators in the algebra generated by $P_1, \dots, P_{d/2}$ and that $\tau^{\varepsilon}_{j\,u}(hg) = \tau^{\varepsilon}_{j\,u}(g)$ for $g \in \Omega_{\varepsilon}$, $h \in H(g)$ (Lemma 5.2.1). Now since $P_1, \dots, P_{d/2}$ are constant when restricted to a single orbit contained in Ω_{ε} (Lemma 4.2.1) we have proved the lemma.

We have defined the functions S_j^ε on Ω_ε by

$$S_j^\varepsilon(g) = \langle g, X_j\rangle - \sum_{u=1}^{d/2} \tau_{j\,u}^\varepsilon(g)\langle g, X_{k_u}\rangle.$$

We take this as the defining relation of S_j^ε when g does not belong to Ω_ε. In this way S_j^ε is a rational function on $\mathfrak{g}^*$. For $j = k_r$, $r = 1, \ldots, d$, we have $S_j^\varepsilon = 0$, and for $k_r < j < k_{r+1}$ we have

$$S_j^\varepsilon(g) = \xi_j + \overline{S}_j^\varepsilon(\xi_1, \ldots, \xi_{k_r}),$$

where $\overline{S}_j^\varepsilon(g)$ is a rational function on $\mathfrak{g}^*$ with denominators in the algebra generated by $P_1, \ldots, P_r$. It follows that we have, using Lemma 5.2.2,

Lemma 5.3.2. *The functions S_j^ε, $1 \le j \le m$, when restricted to a single orbit O contained in Ω_ε, belong to $\mathscr{P}^0_{F(O,\mathscr{F})}(O)$.*

5.4. Construction of canonical coordinates on a single orbit. We now consider the $\mathscr{B}$-chart $\sigma_O^{\mathscr{B}} : O \to \mathbf{R}^d$ of a single orbit $O = Gg$ contained in Ω_ε. Recall that $\sigma_O^{\mathscr{B}}$ is defined as $\sigma_O^{\mathscr{B}} : l \to (\langle l, X_{j_1}\rangle, \ldots, \langle l, X_{j_d}\rangle)$ (see §1.6). In the following we use then notation $\sigma_O^{\mathscr{B}} = (y_{j_1}, \ldots, y_{j_d})$. The inverse of $\sigma_O^{\mathscr{B}}$ is given by $\sigma_O^{\mathscr{B}^{-1}} : (y_{j_1}, \ldots, y_{j_d}) = y \to (R_1(g, y), \ldots, R_m(g, y)) = \sum_{j=1}^m R_j(g, y) l_j$. Recall that $(g, y) \to R_j(g, y)$ is a polynomial function in the variable $y = (y_{j_1}, \ldots, y_{j_d})$ and a rational function in the variable $g \in \mathfrak{g}^*$ with denominators in the algebra genarated by $P_1, \ldots, P_{d/2}$, and that $R_{j_r}(g, y) = y_{j_r}$.

We now define another map $\tau_O^{\mathscr{B}} : O \to \mathbf{R}^d$ in the following manner: $\tau_O^{\mathscr{B}} : l \to (z_{j_1}(l), \ldots, z_{j_d}(l))$, where, for $r = 1, \ldots, d/2$,

$$z_{j_r}(l) = \langle l, X_{j_r}\rangle \quad \text{if } j_r \in K_g$$

and

$$z_{j_r}(l) = S_{j_r}(l) \quad \text{if } j_r \in L_g$$

or, put otherwise,

$$z_{k_r}(l) = \langle l, X_{k_r}\rangle = y_{k_r}(l)$$

and

$$z_{l_r}(l) = S_{l_r}^\varepsilon(l) \quad (\text{or } z_{l_r}(l) = S_{l_r}^\varepsilon(l)\ ).$$

Then $z_{j_1}, \ldots, z_{j_d}$ are polynomial functions on the orbit O, and they belong to $\mathscr{P}^1_{F(O,\mathscr{F})}(O)$, and for $j_r \in L_g$ we have that $z_{j_r} \in \mathscr{P}^0_F(O)$ (Lemma 5.2.2).

Lemma 5.4.1. *The map $\tau_O^{\mathscr{B}}$ is a polynomial chart on O.*

Proof. First we consider $\tau_O^{\mathscr{B}} \circ \sigma_O^{\mathscr{B}^{-1}} : \mathbf{R}^d \to \mathbf{R}^d$. We write $\tau_O^{\mathscr{B}} \circ \sigma_O^{\mathscr{B}^{-1}}(y) = (f_{j_1}(g,y), \dots, f_{j_d}(g,y))$, where $O = Gg$. We have for $1 \le r \le d/2$

$$\begin{aligned} f_{k_r}(g,y) &= y_{k_r}, \\ f_{l_r}(g,y) &= S_{l_r}(R_1(g,y), \dots, R_m(g,y)) \\ &= y_{l_r} + \overline{S}^{\varepsilon}_{l_r}(R_1(g,y), \dots, R_{l_r-1}(g,y)). \end{aligned}$$

We set $\overline{f}_{j_r}(g,y) = \overline{S}^{\varepsilon}_{j_r}(R_1(g,y), \dots, R_{j_r-1}(g,y))$ for $j_r \in L_g$, $1 \le r \le d$. Since $R_j(g,y)$ at most depends on $y_{j_1}, \dots, y_{j_{r-1}}$ for $j < j_r$ we have that

$$\overline{f}_{j_r}(g,y) = \overline{f}_{j_r}(g, y_1, \dots, y_{j_{r-1}}),$$

when $j_r \in L_g$. Let us set $\overline{f}_{j_r} = 0$ for $j_r \in K_g$, $1 \le r \le d$. In this way we have for all $1 \le r \le d$:

$$f_{j_r}(g,y) = y_{j_r} + \overline{f}_{j_r}(g,y) = y_{j_r} + \overline{f}_{j_r}(g, y_{j_1}, \dots, y_{j_{r-1}}).$$

Writing these equations

$$\begin{aligned} z_{j_1} &= y_{j_1}, \\ z_{j_2} &= y_{j_2} + \overline{f}_{j_2}(g, y_{j_1}), \\ z_{j_3} &= y_{j_3} + \overline{f}_{j_3}(g, y_{j_1}, y_{j_2}), \\ &\dots \\ z_{j_d} &= y_{j_d} + \overline{f}_{j_d}(g, y_{j_1}, \dots, y_{j_{d-1}}), \end{aligned}$$

we see by succesive solution that

$$\begin{aligned} y_{j_1} &= z_{j_1}, \\ y_{j_2} &= z_{j_2} - \overline{f}_{j_2}(g, y_{j_1}) = z_{j_2} - \overline{f}_{j_2}(g, z_1), \\ y_{j_3} &= z_{j_3} - \overline{f}_{j_3}(g, y_{j_1}, y_{j_2}) \\ &= z_{j_3} - \overline{f}_{j_3}(g, z_{j_1}, z_{j_2} - \overline{f}_{j_2}(g, z_{j_1})), \\ &\dots \\ y_{j_d} &= z_{j_d} - \overline{f}_{j_d}(g, y_{j_1}, \dots, y_{j_{d-1}}), \end{aligned}$$

and therefore that $\tau_O^{\mathscr{B}} \circ \sigma_O^{\mathscr{B}^{-1}}$ is bijective and its inverse $z \to \sigma_O^{\mathscr{B}} \circ \tau_O^{\mathscr{B}^{-1}}(z) = (h_{j_1}(g,z), \dots, h_{j_d}(g,z)) : \mathbf{R}^d \to \mathbf{R}^d$ is a polynomial function given by

$$y_{j_r} = h_{j_r}(g,z) = z_{j_r} + \overline{h}_{j_r}(g,z) = z_{j_r} + \overline{h}_{j_r}(g, z_{j_1}, \dots, z_{j_{r-1}}),$$

$r = 1, \dots, d$. In other words, $\tau_O^{\mathscr{B}} \circ \sigma_O^{\mathscr{B}^{-1}}$ is a polynomial diffeomorphism. This ends the proof of the lemma.

5.4.2. For later use we note that the functions $(g,y) \to f_{j_r}(g,y)$ and $(g,z) \to h_{j_r}(g,z)$, $g \in \Omega_\varepsilon$, $y \in \mathbf{R}^d$, in a natural way can be extended to be rational functions on $\mathfrak{g}^*$ in the variable g with denominators in the algebra generated by the polynomials $P_1, \dots, P_{d/2}$.

We shall now briefly consider what happens to the canonical symplectic form under the coordinate change $(y_{j_1}, \dots, y_{j_d}) \to (z_{j_1}, \dots, z_{j_d})$. Let us set $\omega_{rs} = \omega_O(\partial/\partial y_{j_r}, \partial/\partial y_{j_s})$ and $\omega'_{rs} = \omega_O(\partial/\partial z_{j_r}, \partial/\partial z_{j_s})$ for $1 \le r, s \le d$, and let $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ be the corresponding matrices. Writing $\boldsymbol{\gamma} = [\partial z_{j_r}/\partial y_{j_s}]_{1\le r,s\le d}$ we have $\boldsymbol{\omega} = {}^t\boldsymbol{\gamma}\boldsymbol{\omega}'\boldsymbol{\gamma}$. Now since $z_{j_r} = f_{j_r}(y) = y_{j_r} + \overline{f}_{j_r}(y_{j_1}, \dots, y_{j_{r-1}})$ we see that $\partial z_{j_r}/\partial y_{j_s} = 1$ for $s = r$, and $\partial z_{j_r}/\partial y_{j_s} = 0$ for $s > r$, and therefore that $\boldsymbol{\gamma}$ is a lower triangular matrix with 1's in the diagonal, hence $\det \boldsymbol{\gamma} = 1$. We conclude that $\det \boldsymbol{\omega} = \det \boldsymbol{\omega}'$. Now we have the following matrix relations $[\{y_{j_r}, y_{j_s}\}]_{1\le r,s\le d} = -\boldsymbol{\omega}^{-1}$, and similarly $[\{z_{j_r}, z_{j_s}\}]_{1\le r,s\le d} = -\boldsymbol{\omega}'^{-1}$. It follows that

$$\begin{aligned} P_{e_m}(l)^2 &= \mathrm{Pf}([\langle l, [X_{j_r}, X_{j_s}]\rangle]_{1\le r,s\le d})^2 \\ &= \mathrm{Pf}([\{y_{j_r}, y_{j_s}\}(l)]_{1\le r,s\le d})^2 = \mathrm{Pf}([\{z_{j_r}, z_{j_s}\}(l)]_{1\le r,s\le d})^2. \end{aligned}$$

We now define the $(d \times d)$-matrix $\mathbf{A}$ in the following way:

$$\begin{aligned} a_{r\,s}(l) &= \{z_{k_r}, z_{k_s}\}, \quad 1 \le r, s \le d/2, \\ a_{r\,s+d/2}(l) &= \{z_{k_r}, z_{\bar{l}_s}\}, \quad 1 \le r, s \le d/2, \\ a_{r+d/2\,s}(l) &= \{z_{\bar{l}_r}, z_{k_s}\}, \quad 1 \le r, s \le d/2, \\ a_{r+d/2\,s+d/2}(l) &= \{z_{\bar{l}_r}, z_{\bar{l}_s}\}, \quad 1 \le r, s \le d/2. \end{aligned}$$

Let us then note that by construction $\{z_{\bar{l}_r}, z_{\bar{l}_s}\} = 0$, hence $a_{r\,s} = 0$ for $d/2 < r, s \le d$, i.e., $\mathbf{A}(l)$ has the form

$$\mathbf{A}(l) = \begin{pmatrix} \mathbf{A}_{11}(l) & \mathbf{A}_{12}(l) \\ -\mathbf{A}_{12}(l) & \mathbf{O} \end{pmatrix}.$$

It follows that $\det \mathbf{A}(l) = (\det \mathbf{A}_{12}(l))^2$. But

$$\det \mathbf{A}(l) = \det([\{z_{j_r}, z_{j_s}\}]_{1\le r,s\le d}(l)) = P_{e_m}(l)^2,$$

hence $\det \mathbf{A}_{12}(l) = \pm P_{e_m}(l)$. In particular $\det \mathbf{A}_{12}(l)$ is constant on each orbit contained in Ω_ε. We define for each $l \in O$ the $(d/2 \times d/2)$-matrix $\mathbf{B}(\ell)$ as

$$\mathbf{B}(l) = \mathbf{A}_{12}(l)^{-1}.$$

Since the entries of $\mathbf{A}_{12}$ are of the form $\{z_{k_r}, z_{\bar{l}_s}\}$, and since $z_{l_s} \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$, we see that the entries of $\mathbf{A}_{12}$ are in $\mathscr{P}^0_{F(O,\mathscr{F})}(O)$. Moreover, since $\mathscr{P}^0_{F(O,\mathscr{F})}(O)$ is a subalgebra and since $\det \mathbf{A}_{12}$ is constant on O we see that the entries of $\mathbf{B}$ are in $\mathscr{P}^0_{F(O,\mathscr{F})}(O)$.

We now define the functions $q_1 = q_1^O, \ldots, q_{d/2} = q_{d/2}^O$ and $\overline{p}_1 = \overline{p}_1^O, \ldots, \overline{p}_{d/2} = \overline{p}_{d/2}^O$ as follows: We set

$$q_r(l) = z_{\overline{l}_r}(l), \quad r = 1, \ldots, d/2,$$

$$\overline{p}_r(\ell) = \sum_{s=1}^{d/2} b_{rs}(l) z_{k_s}(l), \quad r = 1, \ldots, d/2.$$

We note that $q_r^O = q_r^\varepsilon|O$, $r = 1, \ldots, d/2$ (see §5.2). By construction we have $\{q_r, q_s\} = 0$. Moreover, $\{\overline{p}_r, q_s\} = \{\sum_{t=1}^{d/2} b_{rt} z_{k_t}, z_{\overline{l}_s}\} = \sum_{t=1}^{d/2} b_{rt}\{z_{k_t}, z_{\overline{l}_s}\} = (\mathbf{BA}_{12})_{rs} = \delta_{rs}$. We also note that $q_1, \ldots, q_{d/2} \in \mathscr{P}_F^0(O)$ and that $\overline{p}_1, \ldots, \overline{p}_{d/2} \in \mathscr{P}_F^1(O)$.

Define the map $\tau = (x_1, \ldots, x_{d/2})\colon\ G/H \to \mathbf{R}^{d/2}$ by $x_r(sH) = q_r(sg)$, $s \in G$, $1 \le r \le d/2$. This map is well defined by since $q_r \in \mathscr{P}_F^0(O)$.

Lemma 5.4.3. *The map τ defines a polynomial chart on G/H.*

Proof. It follows from Lemma 2.2.6 that the map $\Phi : O \to \mathbf{R}^{d/2} \times G/H$ given by

$$\Phi : l = sg \to (\langle l, X_{k_1}\rangle, \ldots, \langle l, X_{k_{d/2}}\rangle, sH)$$

is a bi-polynomial diffeomorphism, and it follows from Lemma 5.4.1 that the map $O \to \mathbf{R}^d$ given by

$$l \to (\langle l, X_{k_1}\rangle, \ldots, \langle l, X_{k_{d/2}}\rangle, q_1(l), \ldots, q_{d/2})$$

is a bi-polynomial diffeomorphism. But from this we see that the map $G/H \to \mathbf{R}^{d/2}$: $sH \to (q_1(sg), \ldots, q_{d/2}(sg))$ is a bi-polynomial diffeomorphism. This ends the proof of the lemma.

It follows from the previous lemma that $\mathscr{P}_F^0(O)$ are all functions of the form $a(q_1, \ldots, q_{d/2})$, where $a \in \mathscr{P}(\mathbf{R}^{d/2})$.

We now define the map $\overline{\varphi}_O^{\mathscr{B}} : O \to \mathbf{R}^d$ by

$$\overline{\varphi}_O^{\mathscr{B}} : l \to (\overline{p}_1(l), \ldots, \overline{p}_{d/2}(l), q_1(l), \ldots, q_{d/2}(l)).$$

Lemma 5.4.4. *$\overline{\varphi}_O^{\mathscr{B}}$ is a polynomial chart.*

Proof. Consider $\overline{\varphi}_O^{\mathscr{B}} \circ \tau_O^{\mathscr{B}^{-1}} : z \to (\overline{p}(z), q(z))$. We have for $1 \le r \le d/2$:

$$q_r(z) = z_{\overline{l}_r}, \quad \overline{p}_r = \sum_{s=1}^{d/2} b_{rs}(z_{\overline{l}_1}, \ldots, z_{\overline{l}_{d/2}}) z_{k_s},$$

so $\overline{\varphi}_O^{\mathscr{B}} \circ \tau_O^{\mathscr{B}^{-1}}$ is a polynomial function. Moreover, for $1 \le r \le d/2$ we have

$$z_{\overline{l}_r} = q_r,$$

$$z_{k_r} = \sum_{s=1}^{d/2} a_{rs}(l)\overline{p}_s = \sum_{s=1}^{d/2} a_{rs}(q_1, \ldots, q_{d/2})\overline{p}_s.$$

It follows that $\overline{\varphi}_O^{\mathscr{B}} \circ \tau_O^{\mathscr{B}^{-1}}$ is bijective and that $\tau_O^{\mathscr{B}} \circ \overline{\varphi}_O^{\mathscr{B}^{-1}}$ is polynomial. This ends the proof of the lemma.

Before going on, let us note that we have

$$\{\varphi, \psi\} = \sum_{r,s=1}^{d/2} \{\overline{p}_r, \overline{p}_s\} \frac{\partial \varphi}{\partial \overline{p}_r} \frac{\partial \psi}{\partial \overline{p}_s} + \sum_{r=1}^{d/2} \left(\frac{\partial \varphi}{\partial \overline{p}_r} \frac{\partial \psi}{\partial q_r} - \frac{\partial \varphi}{\partial q_r} \frac{\partial \psi}{\partial \overline{p}_r} \right).$$

In particular we have

$$\{\overline{p}_r, \psi\} = \sum_{s=1}^{d/2} \{\overline{p}_r, \overline{p}_s\} \frac{\partial \psi}{\partial \overline{p}_s} + \frac{\partial \psi}{\partial q_r}.$$

If $\psi \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$ it follows that

$$\{\overline{p}_r, \psi\} = \frac{\partial \psi}{\partial q_r}.$$

Lemma 5.4.5. *Let $q_1, \dots, q_{d/2}$ be generators of $\mathscr{P}^0_{F(O,\mathscr{F})}(O)$, and let $\overline{p}_1, \dots, \overline{p}_{d/2}$ be elements in $\mathscr{P}^1_{F(O,\mathscr{F})}(O)$ such that $\{\overline{p}_u, q_v\} = \delta_{uv}$, $1 \le u, v \le d/2$. Then there exists functions $a_1, \dots, a_{d/2} \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$ such that if we set $p_u = \overline{p}_u + a_u$, $1 \le u \le d/2$, then $\{p_u, q_v\} = \delta_{uv}$ and $\{p_u, p_v\} = 0$ for $1 \le u, v \le d/2$.*

Proof. We shall give a recursive way of obtaining such functions a_u: We set $a_{d/2} = 0$. Suppose that $a_{d/2}, \dots, a_j$ have been determined such that $\{p_r, p_s\} = 0$ for $j \le r, s \le d/2$ and $\{p_r, q_s\} = \delta_{rs}$, $1 \le s \le d/2$, $j \le r \le d/2$. We seek an element $a_{j-1} \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$ such that $p_{j-1} = \overline{p}_{j-1} + a_{j-1}$ satisfies $\{p_{j-1}, q_s\} = \delta_{j-1\,s}$ for $1 \le s \le d/2$ and $\{p_{j-1}, p_s\} = 0$ for $j \le s \le d/2$, i.e., such that $\{\overline{p}_{j-1}, q_s\} = \delta_{j-1\,s}$ and $\{\overline{p}_{j-1}, p_s\} = -\{a_{j-1}, p_s\}$ for $j \le s \le d/2$. But the first condition is already satisfied, so the element a_{j-1} has to satisfy $\{\overline{p}_{j-1}, p_s\} = -\{a_{j-1}, p_s\} = -\{a_{j-1}, \overline{p}_s\}$ for $j \le s \le d/2$. Now since a_{j-1} is assumed to be in $\mathscr{P}^0_{F(O,\mathscr{F})}(O)$ we must have $\{a_{j-1}, \overline{p}_s\} = -\partial a_{j-1}/\partial q_s$ which means that it is necessary and sufficient to find a_{j-1} in $\mathscr{P}^0_{F(O,\mathscr{F})}(O)$ such that

$$\frac{\partial a_{j-1}}{\partial q_s} = \{\overline{p}_{j-1}, p_s\}$$

for $s = d/2, \dots, j$. We shall show that this system of equations can be solved. Set $f_s = \{\overline{p}_{j-1}, p_s\}$ for $s = d/2, \dots, j$. First we claim that $f_s \in \mathscr{P}^0_{F(O,\mathscr{F})}(O)$, $j \le s \le d/2$. To see this, note that $\{q_s, f_r\} = -\{q_s, \{p_r, p_{j-1}\}\} = \{p_r, \{\overline{p}_{j-1}\}\} + \{\overline{p}_{j-1}, \{q_s, p_r\}\} = 0$, since either $\{q_s, p_r\} = 0$ or $= 1$ and similarly for $\{\overline{p}_{j-1}, q_s\}$. This means that we can consider $f_1, \dots, f_{d/2}$ as functions of $q_1, \dots, q_{d/2}$, and that we have to solve the equations

$$\frac{\partial a_{j-1}}{\partial q_s} = f_s(q_1, \dots, q_{d/2}), \quad r = d/2, \dots, j.$$

Since we are in a simply connected domain these equations can be solved if and only if we have

$$\frac{\partial f_r}{\partial q_s} = \frac{\partial f_s}{\partial q_r} \quad \text{for } j \le r, s \le d/2,$$

i.e., if and only if

$$\{p_s, f_r\} = \{p_r, f_s\} \quad \text{for } j \le r, s \le d/2.$$

But for $j \le r, s \le d/2$ we have

$$\begin{aligned}\frac{\partial f_r}{\partial q_s} &= \{p_s, f_r\} = \{p_s, \{p_r, \overline{p}_{j-1}\}\} \\ &= -\{p_r, \{\overline{p}_{j-1}, p_s\}\} - \{\overline{p}_{j-1}, \{p_s, p_r\}\} \\ &= \{p_r, \{p_s, \overline{p}_{j-1}\}\} = \{p_r, \{p_s, \overline{p}_{j-1}\}\} = \frac{\partial f_s}{\partial q_r},\end{aligned}$$

and this shows that these equations can be solved. Since the functions f_r are polynomials in $\mathbf{C}[q_1, \dots, q_{d/2}]$, there is an obvious algorithm for for finding a_{j-1}. Note also that the functions a_u are uniquely determined if we require that $p_u(0, \dots, 0) = 0$. This end the proof of the lemma.

We now use Lemma 5.4.5 on the functions $\overline{p}_1, \dots, \overline{p}_{d/2}$ constructed previously to find functions $p_1 = p_1^O, \dots, p_{d/2} = p_{d/2}^O$ satisfying the properties in Lemma 5.4.5. We define $\varphi_O : O \to \mathbf{R}^d$ by

$$\varphi_O : l \to (p_1^O(l), \dots, p_{d/2}^O, q_1^O(l), \dots, q_{d/2}^O(l)).$$

It is immediate that $\varphi_O \circ \overline{p}_O^{-1}$ is a bi-polynomial diffeomorphism. We have thus constructed a polynomial global chart on O.

5.4.6. Using the polynomial global chart $\varphi_O = (p_1, \dots, p_{d/2}, q_1, \dots, q_{d/2})$ we have that $\mathscr{P}_F^0(O)$ consists of functions of the form $a(q)$, where $a \in \mathscr{P}(\mathbf{R}^{d/2})$. It follows that each function $\varphi \in \mathscr{P}_F^1(O)$ can be written uniquely as

$$\varphi = \sum_{r=1}^{d/2} b_r(q)\psi^{X_{k_r}} + b_0(q),$$

where $b_0, b_1, \dots, b_{d/2}$ belong to $\mathscr{P}(\mathbf{R}^{d/2})$ (Proposition 2.2.9). In particular each ψ^X, $X \in \mathfrak{g}$, can be written

$$\psi^X = \sum_{r=1}^{d/2} b_{X,r}(q)\psi^{X_{k_r}} + b_{X,0}(q).$$

Setting $b_{s,r} = b_{X_{j_s},r}$ and using the notation from the beginning of §5.4 this means that we can write each y_{j_s} uniquely as

$$y_{j_s} = \sum_{r=1}^{d/2} b_{s,r}(q)y_{k_r} + b_{s,0}(q),$$

$s = 1, \ldots, d$. Suppose now that $j_s \in L_g$. Then there exists $1 \leq w \leq d/2$ such that $j_s = \bar{l}_w$, and we have

$$\begin{aligned} y_{j_s} &= h_{g, j_s}(z) = z_{j_s} + \bar{h}_{j_s}(g, z_{j_1}, \ldots, z_{j_{s-1}}) \\ &= q_w + \bar{h}_{j_s}(g, z_{j_1}, \ldots, z_{j_{s-1}}), \end{aligned}$$

where we use the notation from the proof of Lemma 5.4.1. Comparing this with the expression above we get that

$$y_{j_s} = y_{\bar{l}_w} = q_w + \sum_{r=1}^{d/2} b_{s,r}(q_1, \ldots, q_{w-1}) y_{k_r} + b'_{0,r}(q_1, \ldots, q_{w-1}),$$

where $b'_{s,0}(q) = -q_w + b_{s,0}(q_1, \ldots, q_{w-1})$.

Let us then use the polynomial canonical coordinates φ_O to construct the representation π as in Theorem 2.7.2. It is then immediate that we can construct elements $v_w \in U(\mathfrak{g}_{\mathbf{C}})$, $w = 1, \ldots, d/2$, such that $d\pi(v_w) = iM_w$, where $M_w f(t) = t_w f(t)$. In fact, set $v_1 = X_{j_1}$ $(= X_{\bar{l}_1})$; then $d\pi(v_1) = iM_1$. Suppose then that we have found $v_1, \ldots, v_{w-1}$, $w < d/2$, such that $d\pi(v_r) = iM_r$ for $1 \leq r \leq w-1$, and let $1 \leq s \leq d$ be such that $\bar{l}_w = j_s$. It then follows from Theorem 2.7.2 that

$$\begin{aligned} d\pi(X_{j_s}) &= iM_w + \sum_{r=1}^{d/2} b_{r,s}(M_1, \ldots, M_{w-1}) d\pi(X_{k_r}) + ib'_{0,r}(M_1, \ldots, M_{w-1}) \\ &= iM_w + \sum_{r=1}^{d/2} b_{r,s}(-id\pi(v_1), \ldots, -id\pi(v_{w-1})) d\pi(X_{k_r}) \\ &\quad + ib'_{0,r}(-id\pi(v_1), \ldots, -id\pi(v_{w-1})) \\ &= iM_w + d\pi\left(\sum_{r=1}^{d/2} b_{r,s}(-iv_1, \ldots, -iv_{w-1}) X_{k_r}\right) \\ &\quad + id\pi\left(b'_{0,r}(-iv_1, \ldots, -iv_{w-1})\right). \end{aligned}$$

Therefore, if we set

$$v_w = X_{j_s} - \sum_{r=1}^{d/2} b_{r,s}(-iv_1, \ldots, -iv_{w-1}) X_{k_r} - ib'_{0,r}(-iv_1, \ldots, -iv_{w-1}),$$

we have that $d\pi(v_w) = iM_w$. It is then clear that we can also find elements $u_r \in U(\mathfrak{g}_{\mathbf{C}})$ such that $d\pi(u_r) = D_r$ (Proposition 2.3.1), and it follows that $d\pi(U(\mathfrak{g}_{\mathbf{C}})) = \mathscr{A}^{d/2}(\mathbf{R}^{d/2})$. This result is of course immediate from Theorem 3.2.1. The point is, however, that here we have proved this result *without* the usual induction-on-the-dimension-of-the-group argument. Having this result it follows easily that the representation π is irreducible.

This means that we have now, for the first time, a noninductive proof of the fact that one can associate a well-defined class of irreducible representations of the group G with each coadjoint orbit. In fact, here we have proved that the representation constructed by means of a Vergne polarization is irreducible, and it follows by a noninductive argument from [10] that *any* representations arising from a polarization satisfies the Kirillov character formula. Since we have proved here that at least one of these representations is irreducible, they all are, and, moreover, they are all equivalent.

We note that we have also given, for the first time, a noninductive proof of the fact that the image by the differential of an irreducible representation associated with an orbit is isomorphic to the Weyl algebra $\mathscr{A}(\mathbf{R}^{d/2})$.

Let us finally remark that there are other basic results from the Kirillov theory that still lack noninductive proofs; the most important of these are the fact that *all* irreducible representaions are associated with orbits, and the fact that the center of the universal enveloping algebra is isomorphic to the algebra of all G-invariant polynomials on $\mathfrak{g}^*$ (where we even have an explicit isomorphism using the symmetrization map).

5.5. The rational functions p^ε. We shall now define the rational functions p_r^ε appearing in Theorem 5.1.1.

First we consider the functions $\overline{p}_r^O$ defined in §5.4. We start by noting that if $\varphi, \psi \in \mathscr{E}(O)$, then

$$\{\varphi, \psi\} = \sum_{u,v=1}^{d} \left(\sum_{k=1}^{m} c_{j_u j_v k} R_k(g, y) \right) \frac{\partial \varphi}{\partial y_{j_u}} \frac{\partial \psi}{\partial y_{j_v}},$$

see §2.1. It follows that if we let $\alpha_{rs}(g, y)$ denote the entries of the matrix $\mathbf{A}_{12}(\sigma_O^{\mathscr{B}^{-1}}(y))$, $O = Gg$, where $\mathbf{A}_{12}(l)$ is defined in §5.4, we get with, the notation from the proof of Lemma 5.4.1, that

$$\alpha_{rs}(g, y) = \sum_{u,v=1}^{d} \left(\sum_{k=1}^{m} c_{j_u j_v k} R_k(g, y) \right) \frac{\partial \mathfrak{f}_{k_r}}{\partial y_{j_u}}(g, y) \frac{\partial f_{\bar{l}_s}}{\partial y_{j_v}}(g, y),$$

and from this it follows that we can extend $\alpha_{rs}(g, y)$ in a natural way to a rational function on $\mathfrak{g}^*$ with denominator in the algebra generated by the polynomials $P_{e_1}, \dots, P_{e_m}$. Let then $\beta_{rs}(g, y)$ denote the entries of the matrix $\mathbf{B}(\sigma_O^{\mathscr{B}^{-1}}(y))$, where also the matrix $\mathbf{B}(l) = \mathbf{A}_{12}(l)^{-1}$ is defined in §5.4. Since $\det \mathbf{B}(l) = P_{e_m}(l)$ for $l \in \Omega_\varepsilon$ we see that we can extend $g \to \beta_{rs}(g, y)$ to a rational function on $\mathfrak{g}^*$ with denominators in the algebra generated by the polynomials $P_{e_1}, \dots, P_{e_m}$. We then set

$$\overline{F}_r(g, y) = \sum_{s=1}^{d/2} \beta_{rs}(g, y) f_{k_s}(g, y), \quad r = 1, \dots, d/2.$$

The functions $(g, y) \to \overline{F}_r(g, y)$ are the polynomials in the variable y and rational functions the variable g with denominators in the algebra generated by the polynomials $P_{e_1}, \dots, P_{e_m}$, and it follows from §5.4 that if $g \in \Omega_\varepsilon$ and $O = Gg$, then

$$\overline{p}_r^O(y) = \overline{F}_r(g, y).$$

Having this result we proceed to consider the functions p_r^O constructed using Lemma 5.4.5. In the proof of Lemma 5.4.5 we constructed $p_d^O, \dots, p_1^O$ be a recursion procedure. Inspecting this procedure, using the result we have already obtained about $\overline{p}_r^O$, and arguing as above we see that there exists functions $F_r(g, y)$ that are polynomial in the variable y and a rational function in the variable g with denominators in the algebra generated by $P_{e_1}, \dots, P_{e_m}$, and such that

$$p_r^O(\sigma_O^{\mathscr{B}^{-1}}(y)) = F_r(g, y),$$

when $g \in \Omega_\varepsilon$, $O = Gg$, $r = 1, \dots, d/2$.

We now define the functions p_r^ε by

$$p_r^\varepsilon(g) = F_r(R_1(g, 0), \dots, R_m(g, 0), \xi_{j_1}, \dots, \xi_{j_d}),$$

where we identify elements $g \in \mathfrak{g}^*$ with their coordinates $\xi_j = \langle g, X_j \rangle$ with respect to the basis in $\mathfrak{g}^*$ dual to $X_1, \dots, X_m$. Clearly the functions p_r^ε, $r = 1, \dots, d/2$, are rational functions with denominators in the algebra generated by $P_{e_1}, \dots, P_{e_m}$, and we have

$$p_r^O = p_r^\varepsilon | O, \quad r = 1, \dots, d/2;$$

in fact, let $l \in O$, and set $y = \sigma_O^{\mathscr{B}}(l)$. Then $\langle l, X_{j_r} \rangle = y_{j_r}$, so

$$\begin{aligned} p_r^O(l) &= F_r(l, y) \\ &= F_r(R_1(l, 0), \dots, R_m(l, 0), y_{j_1}, \dots, y_{j_d}) \\ &= p_r^\varepsilon(l). \end{aligned}$$

We have now found the rational functions p_r^ε and q_r^ε from Theorem 5.1.1, and this ends the proof of Theorem 5.1.1. Let $\mathfrak{g}$ be the 6-dimensional nilpotent Lie algebra with a basis $\mathscr{B} : X_1, \dots, X_6$ satisfying the following nonvanishing commutation relations (we only give $[X_i, X_j]$ for $i > j$):

$$\begin{gathered} [X_6, X_5] = X_4, \ [X_6, X_4] = X_3, \ [X_6, X_3] = X_2, \\ [X_5, X_4] = X_2, \ [X_5, X_2] = -X_1, \ [X_4, X_3] = X_1. \end{gathered}$$

We only carry out the calculations in the first stratum. The other strata are handled similarly. The canonical coordinates (p_1, p_2, q_1, q_2) have been calculated using the algorithm from §5.

1*st stratum*: $\xi_1 \neq 0$:

The 5-tuple of jump indices: $\varepsilon = (\varnothing, \varnothing, \varnothing, \{3, 4\}, \{2, 3, 4, 5\}, \{2, 3, 4, 5\})$

The dimension of orbits: $d = 4$.

The jump indices:

$j_1 = 2\,,\; j_2 = 3\,,\; j_3 = 4\,,\; j_4 = 5\,,$
$k_1 = 4\,,\; k_2 = 5\,,$
$l_1 = 3\,,\; l_2 = 2\,,$
$\bar{l}_1 = 2\,,\; \bar{l}_2 = 3\,.$

The polynomials P_{e_j}:

$$P_{e_3} = \xi_1\,,$$
$$P_{e_4} = -\xi_1^2\,.$$

The functions $\psi^{X_j} = R_j$ expressed in the coordinates (y_2, y_3, y_4, y_5):

$$R_1(g, y) = y_1 = \xi_1\,,$$
$$R_2(g, y) = y_2\,,$$
$$R_3(g, y) = y_3\,,$$
$$R_4(g, y) = y_4\,,$$
$$R_5(g, y) = y_5\,,$$
$$R_6(g, y) = y_6 = \xi_6 + \frac{1}{6\xi(1)^2}(-6\xi_1\xi_2\xi_4 + 3\xi_1\xi_3^2 - 2\xi_2^3 + 6\xi_1 y_2 y_4 - 3\xi_1 y_3^2 + 2y_2^3)\,.$$

The rational functions $(p_1^\varepsilon, p_2^\varepsilon, q_1^\varepsilon, q_2^\varepsilon)$:

$$p_1^\varepsilon = \frac{1}{\xi_1^2}(-\xi_1\xi_5 - \xi_2\xi_3)\,,$$
$$p_2^\varepsilon = \frac{1}{\xi_1}\xi_4\,,$$
$$q_1^\varepsilon = \xi_2\,,$$
$$q_2^\varepsilon = \xi_3\,.$$

The canonical coordinates (p_1, p_2, q_1, q_2) expressed in the coordinates (y_2, y_3, y_4, y_5):

$$p_1 = \frac{1}{\xi_1^2}(-\xi_1 y_5 - y_3 y_2)\,,$$
$$p_2 = \frac{1}{\xi_1} y_4\,,$$
$$q_1 = y_2\,,$$
$$q_2 = y_3\,.$$

The functions ψ^{X_j} expressed in the canonical coordinates (p_1, p_2, q_1, q_2):

$$y_1 = \xi_1\,,$$
$$y_2 = q_1\,,$$

$$y_3 = q_2,$$
$$y_4 = \xi_1 p_2,$$
$$y_5 = \frac{1}{\xi_1}(-\xi_1^2 p_1 - q_2 q_1),$$
$$y_6 = \frac{1}{6\xi_1^2}(6\xi_1^2\xi_6 + 6\xi_1^2 q_1 p_2 - 6\xi_1\xi_2\xi_4 + 3\xi_1\xi_3^2 - 3\xi_1 q_2^2 - 2\xi_2^3 + 2q_1^3).$$

The representations calculated from the formula in Theorem 3.2.1 using the global canonical coordinates (p_1, p_2, q_1, q_2):

$$d\pi(X_1) = i\xi_1,$$
$$d\pi(X_2) = it_1,$$
$$d\pi(X_3) = it_2,$$
$$d\pi(X_4) = \xi_1 \frac{\partial}{\partial t_2},$$
$$d\pi(X_5) = \frac{1}{\xi_1}(-\xi_1^2 \frac{\partial}{\partial t_1} - it_1 t_2),$$
$$d\pi(X_6) = \frac{1}{6\xi_1^2}(6\xi_1^2\xi_6 + 6\xi_1^2 t_1 \frac{\partial}{\partial t_2} - 6\xi_1\xi_2\xi_4 + 3\xi_1\xi_3^2 - 3i\xi_1 t_2^2 - 2\xi_2^3 + 2it_1^3).$$

The elements $u_1, u_2, v_1, v_2 \in U(\mathfrak{g}_{\mathbf{C}})$ defined in §3.2, cf. Theorem 3.2.1:

$$u_1 = \frac{1}{\xi_1^2}(\xi_1 X_5 + iX_3 X_2),$$
$$u_2 = \frac{1}{\xi_1} X_4,$$
$$v_1 = X_2,$$
$$v_2 = X_3.$$

Added in proof. In a recent paper by P. Bonnet, *Paramétrisation du dual d'une algèbre de Lie nilpotente*, Ann. Inst. Fourier (Grenoble) **38** (1988), 169–197, results similar to those of §5 in the present paper have been obtained.

REFERENCES

1. D. Arnal, J. C. Cortet, P. Molin and G. Pinczon, *Covariance and geometrical invariance in ∗-quantization*, J. Math. Phys. **24** (1983), 276–283.
2. L. Corwin, F. P. Greenleaf and R. Penney, *A general character formula for irreducible projections in L^2 of a nilmanifold*, Math. Ann. **225** (1977), 21–32.
3. J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars, Paris, 1974.
4. J. Dixmier, *Représentations irréductibles des algèbres de Lie nilpotents*, An. Acad. Brasil Ciênc. **53** (1963), 491–519.
5. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), 57–110.
6. A. I. Malcev, *On a class of homogeneous spaces*, Amer. Math. Soc. Transl. (1) **39** (1951).
7. N. V. Pedersen, *On the infinitesimal kernel of irreducible representations of nilpotent Lie groups*, Bull. Soc. Math. France **112** (1984), 423–467.

8. ——, *Characters of solvable Lie groups*, Harmonische Analyse und Darstellungstheorie topologischer Gruppen, Tagungsbericht 33/1985, Mathematisches Forschungsinstitut, Oberwolfach, 1985.

9. ——, *On the symplectic structure of coadjoint orbits of (solvable) Lie groups and applications.* I, Math. Ann. **281** (1988), 633–669.

10. L. Pukanszky, *On the characters and the Plancherel formula of nilpotent groups*, J. Funct. Anal. **1** (1967), 255–280.

11. M. Vergne, *La structure de Poisson sur l'algèbre symmétrique d'une algèbre de Lie nilpotente*, Bull. Soc. Math. France **100** (1972), 301–335.

Mathematical Institute, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark